Modular Spin Characters of Symmetric Groups

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Lukas Alexander Maas
geboren in Moers

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Universität Duisburg–Essen

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Gutachter: Prof. Dr. Wolfgang Lempken
PD Dr. Jürgen Müller
Prof. Dr. Klaus Lux

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für Julia

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Introduction

The present work is concerned with concrete computations of modular spin characters of symmetric and alternating groups. More precisely, we calculate all p-modular spin characters of the symmetric group S_n and the alternating group A_n for $n \in \{14, \dots, 18\}$ and $p \in \{3, 5, 7\}$. Hence we obtain the p-modular decomposition numbers of projective representations of S_n and A_n for the specified values of n and p.

A projective representation of a finite group X on a finite dimensional $\mathbb{C}$-vector space V may be seen as a map $\rho : X \to GL(V)$ such that $1^\rho = \mathrm{id}_V$ and $(xy)^\rho = r(x, y)x^\rho y^\rho$ for all $x, y \in X$ and a corresponding factor set $r : X \times X \to \mathbb{C}^\times$. Thus ρ induces a homomorphism from X to the projective linear group $PGL(V) = GL(V)/\mathbb{C}^\times \mathrm{id}_V$. Projective representations $\rho : X \to GL(V)$ and $\rho' : X \to GL(V')$ are equivalent if there is a $\mathbb{C}$-isomorphism $\sigma : V \to V'$ and a map $s : X \to \mathbb{C}^\times$ such that $\sigma^{-1}x^\rho\sigma = s(x)x^{\rho'}$ for all $x \in X$. In that case the corresponding factor sets r and r' are said to be equivalent, which means that $r'(x, y) = s(x)s(y)s(xy)^{-1}r(x, y)$ for all $x, y \in X$ and some map $s : X \to \mathbb{C}^\times$. The equivalence classes of factor sets of X form a finite abelian group $M(X) \cong H^2(X, \mathbb{C}^\times)$, the Schur multiplier of X. Schur [65, 66] developed a point of view from which the study of projective representations of X can be reduced to the study of linear representations of certain covering groups $\tilde{X}$ of X. He proved that there exists a representation group $\tilde{X}$ of X, that is, a central extension

$$1 \longrightarrow Z \longrightarrow \tilde{X} \xrightarrow{\pi} X \longrightarrow 1$$

with $M(X) \cong Z$ such that every projective representation ρ of X can be lifted to a linear representation $\tilde{\rho}$ of $\tilde{X}$ so that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\rho}} & GL(V) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\rho} & PGL(V) \end{array}$$

Specifically, for $X_n \in \{S_n, A_n\}$ Schur [67] proved that $M(X_n) \cong \mathbb{Z}_2$, with the exception of $M(A_6) \cong M(A_7) \cong \mathbb{Z}_6$ and $M(X_n) = 1$ for $n \leqslant 3$. Moreover, he showed that up to isomorphism there exist two representation groups of S_n ($n \geqslant 4$), the double covers

$$\tilde{S}_n := \langle\ z, t_1, \ldots, t_{n-1}\ :\ z^2 = 1,\ t_i^2 = (t_i t_{i+1})^3 = z,\ (t_j t_k)^2 = z\ \rangle$$

where the relations are imposed for all admissable i, j, k with $|j - k| > 1$, and likewise

$$\hat{S}_n := \langle\ z, \hat{t}_1, \ldots, \hat{t}_{n-1}\ :\ z^2 = 1,\ \hat{t}_i^2 = (\hat{t}_i \hat{t}_{i+1})^3 = 1,\ (\hat{t}_j \hat{t}_k)^2 = z,\ z\hat{t}_i = \hat{t}_i z\ \rangle.$$

Although $\tilde{S}_n$ and $\hat{S}_n$ are non-isomorphic groups for $n \neq 6$, their group algebras over a field F containing a primitive fourth root of unity ζ are isomorphic: the central orthogonal idempotents $z_\pm := (1 \pm z)/2$ induce the decompositions $F\tilde{S}_n = z_+F\tilde{S}_n \oplus z_-F\tilde{S}_n$ and $F\hat{S}_n = z_+F\hat{S}_n \oplus z_-F\hat{S}_n$. Then $z_+F\tilde{S}_n \cong FS_n \cong z_+F\hat{S}_n$ and $z_-F\tilde{S}_n \cong z_-F\hat{S}_n$ via $\iota : z_-F\tilde{S}_n \to z_-F\hat{S}_n$, $z_-t_i \mapsto z_-\zeta\hat{t}_i$. Hence we may restrict our attention to $F\tilde{S}_n$.
The natural epimorphism $\pi : \tilde{S}_n \to S_n$, $t_i \mapsto (i, i+1)$ with kernel $Z(\tilde{S}_n) = \langle z \rangle \cong \mathbb{Z}_2$ yields the double cover $\tilde{A}_n := A_n^{\pi^{-1}}$ of A_n as the preimage of A_n under π. Irreducible faithful representations of $\tilde{X}_n$, that is, irreducible representations of $z_-F\tilde{S}_n$, are called spin representations and a similar "spin" terminology is used for all related faithful objects, to set them apart from the non-faithful objects that belong esssentially to X_n. Observe that every irreducible spin representation maps the central element z to -1.
Spin representation theory of symmetric groups was originated by Schur [67] who determined the representation groups of S_n and A_n, and produced a classification of the ordinary spin representations including a description of their characters. Later Schur's work was revised and extended by Morris [48–50], Jozefiak [30], Stembridge [68], Hoffman–Humphreys [21], and others. Morris [51] proved a reduction rule similar to the Murnaghan–Nakayama formula for symmetric groups that allows to compute single values of ordinary spin characters efficiently. Noeske [56] made the ordinary character tables generically available in the computer algebra system GAP [18].
A large part of what is known in modular spin representation theory consists of the description of spin blocks. Humphreys [23] settled Morris' conjecture on the block distribution of ordinary spin representations via p-bar cores, Cabanes [11] determined the structure of the defect groups. Olsson's studies [59, 60] provide our strategy of handling spin blocks of $\tilde{S}_n$ and $\tilde{A}_n$ simultaneously. Motivated by explicit decomposition numbers computed by Morris and Yaseen [52, 71], Bessenrodt–Morris–Olsson [3] produced a shape result about 3-modular spin decomposition matrices including combinatorial labels. Similar results were obtained by Andrews–Bessenrodt–Olsson [1, 2] in charac-

teristic 5 but their choice of labels turned out to be unsatisfactory for larger characteristics. Eventually, Brundan–Kleshchev [8, 9] provided two 'distinct' classifications of the p-modular spin representations following a conjecture by Leclerc–Thibon [36] on the appropriate labels. While the first approach [9] uses Sergeev duality in parallel with Schur–Weyl duality, [8] follows the Lie-theoretical context of the former development of the p-modular representation theory of symmetric groups in connection with the affine Kac–Moody algebra of type $A_{p-1}^{(1)}$. Specifically, it turns out that the p-modular (socle) branching graph of spin representations coincides with the crystal graph of the basic module of the twisted affine Kac–Moody algebra of type $A_{p-1}^{(2)}$. Only recently, Kleshchev–Shchigolev [33] unified both classification results.

Still a key problem in the representation theory of (spin) symmetric groups is to find a description of p-modular decomposition numbers. Even the dimensions of irreducible (spin) modules in characteristic p are not known in general. Müller [54] has determined the Brauer trees of spin blocks of defect 1, which in principle determine the decomposition matrices. In particular, this explains why we do not consider any characteristic larger than 7 here. Nevertheless, for the sake of completeness we collect all Brauer trees that belong to the examples of this text in Appendix A.

For spin blocks of higher defect, general results on decomposition numbers are rare, possibly because "the numerical data on decomposition matrices for spin characters are very restricted" (p. 147 in [3]). The decomposition matrices for spin characters of $\tilde{X}_n$ for $n \leqslant 13$ date back to [27, 52, 71] and the Modular Atlas community [47]. Thus one aim of this work is to extend the available data. We believe that our results are substantial enough to put forward new conjectures. Some other purpose is given by Donovan's conjecture that there are only finitely many Morita-equivalence classes for spin blocks with a prescribed defect group. This has been settled by Kessar [31], so now their classification is desirable. In addition to the concrete results, this text collects suitable techniques that allow the computation of further decomposition numbers of $\tilde{X}_n$. Of course, one should bear in mind that limits for most of these methods are naturally set by computational resources.

We sketch briefly the contents of the following sections. Section 1 sets the ground for our computations. We introduce a standard setting that allows a uniform definition of Brauer characters, and recall some general concepts from modular representation theory such as projective characters and decomposition numbers. Clifford's theory on index-2 subgroups and the Witt–Berman theorem on splitting fields for finite groups conclude the exposition.

In Section 2 we present methods from modular character theory for computing decomposition numbers. In particular, we introduce the notion of basic sets and revise the FBA algorithm that is used to find them. All these methods were originally accumulated in the MOC system [19] by Hiss, Jansen, Lux, and Parker. The present work relies on our own implementation in GAP.

When calculations based on characters stop, often modules come into play. In Section 3 we explain some features of Parker's MeatAxe [61], introduce the theoretical concept of condensation and consider a practical method for condensing tensor products of modules. Further methods are presented, for example, a simple way of computing symmetry bases for modules under restriction to (reasonably small) abelian condensation subgroups. We remark that most of the material of Sections 1–3 is not specifically designed for any particular type of finite group.

Section 4 moves towards the representation theory of $\tilde{S}_n$. Before the classification theorems on spin representations and the branching rules are stated, some basic notions from supermathematics are introduced. In some way it is a clever move to consider superalgebras in spin representation theory. This in fact neglects the problems of splitting pairs of associate representations which often demands considerable computational effort in practice.

In Section 5 we dig into the group $\tilde{S}_n$. The conjugacy classes of $\tilde{S}_n$ are fixed by means of certain standard representatives that enable us to determine consistent class fusions of various subgroups used for inducing characters. We consider the important class of Young subgroups and develop a practical method for producing their (modular) character tables. Some rather tedious computations inside $\tilde{S}_n$ are needed to prove a rule that determines the class fusion of Young subgroups, see (5.1.6), which is crucial in our context. Finally, we state some explicit results about decomposition numbers of spin representations, namely Olsson's theorem [60] on the crossover correspondence between spin blocks of $\tilde{S}_n$ and $\tilde{A}_n$, and Wales' decomposition theorems [69] of the basic spin representations. Basic spin representations play in fact a special role in spin representation theory. In Section 6 we recapitulate their construction over fields of arbitrary characteristic [45].

The Sections 7–21 compile the proofs of our main results named in the beginning of this introduction. Based on the logfiles of our computer programs, each proof is presented in a step by step manner to facilitate the reproduction for the reader. At the end of each of these sections, the corresponding decomposition matrices are printed. The combinatorial labels and the character degrees are given too. The way we match modular spin characters with their labels as well as our $\pm$-convention is explained in Section 4.3. Further conventions on the labelling of characters are to be found in Section 5.3 and the preliminaries to Section 7. For the layout of the decomposition matrices, we did not choose any particular ordering other than the most neutral lexicographical one. Perhaps this encourages the reader to look for patterns that will ultimately reveal some more decomposition numbers.

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1 The setting

We introduce a "standard setting" that will underlie our subsequent computations and give some known results. For $X \in \{\mathbb{Z}, \mathbb{Q}\}$, we denote $X^+ := \{x \in X : x \geqslant 0\}$.

Finite fields

An explicit field arithmetic is crucial for working with concrete modular representations. The following definition of a finite field with p^n elements that we adhere to is used consistently by GAP [18] and the C–MeatAxe programs [62], see [28, 37, 55].

Let p be a rational prime. We fix the representatives $0, 1, \ldots, p-1$ of the cosets of integers modulo p to denote the elements of $\mathrm{GF}(p) = \mathbb{Z}_p$, and we agree on the ordering $0 < 1 < \ldots < p-1$. The smallest primitive element of $\mathrm{GF}(p)$ is then denoted by ζ_p. Now suppose that $n > 1$. An ordering on the polynomials of degree n over $\mathrm{GF}(p)$ is defined as follows: if $f = \sum_{i=0}^{n} f_i x^i$ and $g = \sum_{i=0}^{n} g_i x^i$, then $f < g$ if and only if $(-1)^{n-k} f_k < (-1)^{n-k} g_k$ where k is the maximal index such that $f_k \neq g_k$. With respect to this ordering, the *Conway polynomial* for $\mathrm{GF}(p^n)$ is defined recursively as the smallest monic, primitive polynomial $f_{p,n}$ of degree n such that for every proper divisor m of n the $(p^n-1)/(p^m-1)$th power of a root of $f_{p,n}$ is a root of $f_{p,m}$. Such a polynomial always exists, see [55]. Thus we can define

$$\mathrm{GF}(p^n) = \mathrm{GF}(p)[x]/(f_{p,n})$$

with designated primitive element $\zeta_{p,n} := x + (f_{p,n})$. Note that $f_{p,n}$ is primitive, hence every root of $f_{p,n}$ is a generator of $\mathrm{GF}(p^n)^\times$. Consequently, if $m \mid n$ and ω is a root of $f_{p,n}$, and $d := (p^n-1)/(p^m-1)$, then

$$\mathrm{GF}(p^m)^\times = \langle\, \omega^d \,\rangle \leqslant \mathrm{GF}(p^n)^\times$$

ensures the embedding of $\mathrm{GF}(p^m)$ as a subfield of $\mathrm{GF}(p^n)$. We identify $\zeta_{p,n}^d = \zeta_{p,m}$.

The modular system

Let G be a finite group of exponent m and let p be a rational prime. Write $m = p^a b$,

where a and b are suitable integers with $p \nmid b$. Let $\xi_m := \exp(2\pi i/m) \in \mathbb{C}$. Since we can find suitable integers r, s (uniquely determined modulo m) such that $rp^a + sb = 1$, we may define the p-part $\xi_{p^a} := \xi_m^{sb}$ and the p'-part $\xi_b := \xi_m^{rp^a}$ of ξ_m. Then ξ_{p^a} has order p^a, ξ_b has order b, and $\xi_m = \xi_{p^a}\xi_b = \xi_b\xi_{p^a}$. We choose $n \in \mathbb{N}$ minimally subject to the condition $p^n \equiv 1 \bmod b$ and set $\xi_{n,p} := \exp(2\pi i/(p^n - 1)) \in \mathbb{C}$.
Let $f := GF(p^n)$ with primitive element $\zeta_{n,p}$.[#1] Declaring $\hat{\zeta}_{n,p} := \xi_{n,p}$, each element $\zeta_{n,p}^k \in f^\times$ may be lifted to $\hat{\zeta}_{n,p}^k = \xi_{n,p}^k \in \mathbb{Z}[\xi_{n,p}]$. Conversely, $\bar{\xi}_{n,p} := \zeta_{n,p}$ induces a ring epimorphism $\bar{\ } : \mathbb{Z}[\xi_{n,p}] \to f$. Note that $\xi_{n,p}^{(p^n-1)/b} \in \mathbb{Q}(\xi_m)$ and $\bar{\xi}_b \in f$ are primitive bth roots of unity. In the following, we assume that $(\mathcal{F}, \mathcal{O}, f, \eta)$ is a standard p-modular system for G as in [42, (4.2.10)], that is, $\mathcal{F}$ is the field of fractions of the completion $\mathcal{O}$ of a valuation ring in $\mathbb{Q}(\xi_m)$ and $\eta : \mathcal{O} \to f$ is a ring epimorphism with $\eta|_{\mathbb{Z}[\xi_b]} = \bar{\ }|_{\mathbb{Z}[\xi_b]}$. In particular, $\xi_m \in \mathcal{F}$ and $\bar{\xi}_b \in f$ ensure that both $\mathcal{F}$ and f are splitting fields for G and all of its subgroups.

Brauer characters

Let $\mathrm{Irr}(G)$ denote the set of ordinary characters of G afforded by a complete set of representatives of the isomorphism classes of irreducible $\mathcal{F}G$-modules. Then $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathcal{C}(G, \mathbb{C})$, the complex vector space of class functions from G to $\mathbb{C}$, with respect to the scalar product

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

If W is an fG-module corresponding to a representation $\delta : G \to GL_d(f)$, we define its *Brauer character* as follows. Let $G_{p'}$ denote the set of p-regular elements of G (that is, elements whose order is not divisible by p), and choose representatives $g_1, \dots, g_l$ of the G-conjugacy classes contained in $G_{p'}$ (the p-*regular classes* of G). For each $i \in \{1, \dots, l\}$, we lift the eigenvalues $\epsilon_1, \dots, \epsilon_d \in \langle \zeta_{n,p} \rangle$ of g_i^δ to elements $\hat{\epsilon}_1, \dots, \hat{\epsilon}_d \in \mathbb{Z}[\xi_{n,p}]$ and set

$$\phi(g_i) := \sum_{j=1}^{d} \hat{\epsilon}_j.$$

Since conjugate matrices have the same multiset of eigenvalues, this defines a class function $\phi : G_{p'} \to \mathbb{C}$, the p-*modular character* afforded by W. As our p-modular system is fixed, we shall often call ϕ simply a Brauer character of G. By definition, ϕ

[1] In GAP, we have $\xi_{n,p} = \mathtt{E}(p^n - 1)$ and $\zeta_{p,n} = \mathtt{Z}(p^n)$.

is irreducible if and only if W is irreducible. We let $\mathrm{IBr}(G)$ denote the set of Brauer characters of G afforded by a complete set of pairwise non-isomorphic irreducible fG-modules. As is known, $|\mathrm{Irr}(G)|$ is the total number of conjugacy classes of G, whereas $|\mathrm{IBr}(G)|$ is the number l of those contained in $G_{p'}$.

Suppose that $g \in G_{p'}$ has order n and all entries of the matrix g^δ are contained in some field $F \leqslant f$. Under the requirement that all irreducible factors of the polynomial $x^n - 1$ over F are known, the following method is used in practice to compute the Brauer character value $\phi(g)$ of g^δ. Let $x^n - 1 = m_1^{n_1} \cdots m_r^{n_r}$ with monic irreducible factors $m_i \in F[x]$. Each m_i is the minimal polynomal over F of $d_i := \deg m_i$ distinct potential eigenvalues $\epsilon_{i_1}, \dots, \epsilon_{i_{d_i}}$ of g^δ. Now any of them is actually an eigenvalue of g^δ with multiplicity e_i if and only if all of them are eigenvalues of g^δ with multiplicity e_i. Set $\nu_i := \hat{\epsilon}_{i_1} + \cdots + \hat{\epsilon}_{i_{d_i}}$ and note that the knowledge of m_i and ν_i is independent of g^δ. Thus it suffices to compute the multiplicity $e_i = \dim \ker m_i(g^\delta)/d_i$ for $i \in \{1, \dots, r\}$. Then

$$\phi(g) = \sum_{i=1}^{r} e_i \nu_i.$$

Let $g \in G$ have p-part g_p and p'-part $g_{p'}$ (defined analogously as above for ξ_m). There is only one eigenvalue of g_p^δ, namely 1, and $g_{p'}$ has the same multiset of eigenvalues as g. In particular, if $g_{p'}$ and g_i are conjugate, then $\overline{\phi(g_i)} = \overline{\phi(g_{p'})}$ is just the trace $\mathrm{tr}_W(g)$ of g^δ on W.#2

Suppose that we know ϕ. If we are interested in δ, usually it is not necessary to work over the (possibly large) field f. In fact, δ can be realized over a field $F \leqslant f$ if and only if $\mathrm{tr}_W(g_i) \in F$ for all $i \in \{1, \dots, l\}$, see [24, (9.14)]. Thus we can determine the size of the smallest such field F from ϕ.#3

Decomposition numbers

Let V be an $\mathcal{F}G$-module affording the character χ. Choosing an $\mathcal{O}$-form $M \leqslant V$, the Brauer character of the p-modular reduction $\overline{V}_M$ of V with respect to M,

$$\chi|_{p'} := \chi|_{G_{p'}} = \sum_{\phi \in \mathrm{IBr}(G)} d_{\chi\phi} \phi,$$

[2]This "Frobenius character" value of g^δ is given in GAP by `FrobeniusCharacterValue(`$\phi(g_i), p$`)`.
[3]Here the corresponding GAP command is `SizeOfFieldOfDefinition(`ϕ, p`)`.

decomposes non-negatively over $\mathrm{IBr}(G)$. The coefficient $d_{\chi\phi}$ of the irreducible Brauer character ϕ afforded by an fG-module W_ϕ is exactly the number of composition factors of $\overline{V}_M$ that are isomorphic to W_ϕ. Moreover, the numbers $d_{\chi\phi}$ are independent of the $\mathcal{O}$-form M. Sometimes we shall simply call $d_{\chi\phi}$ the multiplicity of W_ϕ in a p-modular reduction $\overline{V}$ of V, and if $d_{\chi\phi} \neq 0$, we call W_ϕ an irreducible constituent of $\overline{V}$ or say that W_ϕ is contained in $\overline{V}$. The same terminology is used for the characters ϕ and $\chi|_{p'}$. Decomposing $\chi|_{p'}$ for $\chi \in \mathrm{Irr}(G)$ in terms of $\mathrm{IBr}(G)$ induces the decomposition map

$$\Delta : \langle \mathrm{Irr}(G)\rangle_{\mathbb{Z}} \to \langle \mathrm{IBr}(G)\rangle_{\mathbb{Z}}.$$

Collecting the (p-modular) *decomposition numbers* $d_{\chi\phi}$ for all $\chi \in \mathrm{Irr}(G)$, we obtain the (p-modular) *decomposition matrix* $D := [d_{\chi\phi}]_{\chi\in\mathrm{Irr}(G),\phi\in\mathrm{IBr}(G)}$ of G. It is known that D has full rank l and all of its elementary divisors are 1, thus Δ is surjective (see [16, (6.7)]).

Projective characters

The *projective indecomposable character* corresponding to $\phi \in \mathrm{IBr}(G)$ is defined as

$$\Phi_\phi := \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\phi}\chi.$$

Let $\mathrm{IPr}(G)$ denote the set of projective indecomposable characters of G. The non-zero elements of $\langle \mathrm{IPr}(G)\rangle_{\mathbb{Z}^+}$ are called projective characters.

Let W_ϕ denote an irreducible fG-module that affords $\phi \in \mathrm{IBr}(G)$, and let P_ϕ denote the corresponding projective indecomposable fG-module with $\mathrm{soc}(P_\phi) \cong W_\phi \cong \mathrm{hd}(P_\phi)$ (which is uniquely determined by this property, up to isomorphism). Then there is a projective $\mathcal{O}G$-module $\hat{P}_\phi$ such that $\overline{\hat{P}}_\phi = P_\phi$, and again, $\hat{P}_\phi$ is uniquely determined (up to isomorphism) by this property. Furthermore, $\hat{P}_\phi \otimes_{\mathcal{O}G} \mathcal{F}$ is an $\mathcal{F}G$-module with character $\Phi_\phi \in \mathrm{IPr}(B)$. If V_χ is an irreducible $\mathcal{F}G$-module affording $\chi \in \mathrm{Irr}(G)$, then the multiplicity $d_{\chi\phi}$ of W_ϕ in a p-modular reduction $\overline{V}_\chi$ of V_χ is equal to the multiplicity of V_χ as a constituent of $\hat{P}_\phi \otimes_{\mathcal{O}G} \mathcal{F}$. This is known as Brauer reciprocity. If W is an fG-module with Brauer character $\varphi = \sum_{\phi\in\mathrm{IBr}(G)} m_\phi\phi$, then W involves W_ϕ with multiplicity m_ϕ. We define the projective character corresponding to φ as $\Phi_\varphi := \sum_{\phi\in\mathrm{IBr}(G)} m_\phi\Phi_\phi$.

Since every projective indecomposable character Φ_ϕ vanishes on $G \setminus G_{p'}$, see [16, (6.9)], it is sometimes convenient to identify Φ_ϕ with $\Phi_\phi|_{p'}$. In this way, $\mathrm{IBr}(G)$ and $\mathrm{IPr}(G)$

are dual bases of $\mathcal{C}(G_{p'}, \mathbb{C})$ with respect to the restriction of $\langle\cdot\,,\cdot\rangle$ to $G_{p'}$. Thus

$$\langle \phi, \Phi_\psi \rangle = \frac{1}{|G|} \sum_{g \in G_{p'}} \phi(g)\Phi_\psi(g^{-1}) = \delta_{\phi,\psi}$$

(Kronecker delta $\delta_{\phi,\psi}$) for $\phi \in \mathrm{IBr}(G)$ and $\Phi_\psi \in \mathrm{IPr}(G)$. Equally well, it is sometimes useful to extend a class function ϕ on $G_{p'}$ to class function on G by setting $\phi(g) := \phi(g_{p'})$ for $g \in G \setminus G_{p'}$.

(1.1) Proposition. Let ι and κ be bijections on $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$, respectively, such that $d_{\chi^\iota\phi^\kappa} = d_{\chi\phi}$ for all $\chi \in \mathrm{Irr}(G)$, $\phi \in \mathrm{IBr}(G)$. Then $(\Phi_\phi)^\iota = \Phi_{\phi^\kappa}$ for all $\phi \in \mathrm{IBr}(G)$.
In particular, for a Brauer character $\varphi = \sum_{\phi\in\mathrm{IBr}(G)} m_\phi\phi$ the following are equivalent:

(1) $\varphi^\kappa = \varphi$,

(2) $m_{\phi^\kappa} = m_\phi$ for all $\phi \in \mathrm{IBr}(G)$,

(3) $(\Phi_\varphi)^\iota = \Phi_\varphi$.

Proof. Extend ι and κ linearly onto $\langle\mathrm{Irr}(G)\rangle_{\mathbb{Z}}$ and $\langle\mathrm{IBr}(G)\rangle_{\mathbb{Z}}$, respectively. We have

$$(\Phi_\phi)^\iota = \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\phi}\chi^\iota = \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi^\iota\phi^\kappa}\chi^\iota = \Phi_{\phi^\kappa}. \qquad \blacksquare$$

Blocks

Denote by g^+ the sum over the G-conjugacy class g^G in $\mathcal{F}G$. For every $\chi \in \mathrm{Irr}(G)$, there is a central character $\omega_\chi : Z(\mathcal{F}G) \to \mathbb{C}$ with $\omega_\chi(g^+) = \chi(g)|g^G|/\chi(1) \in \mathbb{Z}[\xi_b]$ for $g \in G$. We write $\chi \sim \psi$ for $\chi, \psi \in \mathrm{Irr}(G)$ if in $\mathbb{Z}[\xi_b]$ we have (see [42, (4.4.8)]) $\omega_\chi(g^+) \equiv \omega_\psi(g^+) \bmod p$ for all $g \in G_{p'}$. Then $\sim$ is an equivalence relation on $\mathrm{Irr}(G)$. Now a p-*block* of G is a subset $B \subseteq \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ such that

(i) $\mathrm{Irr}(B) := B \cap \mathrm{Irr}(G)$ is an equivalence class under $\sim$, and

(ii) $\mathrm{IBr}(B) := B \cap \mathrm{IBr}(G) = \{\, \phi \in \mathrm{IBr}(G) : d_{\chi\phi} \neq 0$ for some $\chi \in \mathrm{Irr}(B) \,\}$.

We set $\mathrm{IPr}(B) := \{\, \Phi_\phi \in \mathrm{IPr}(G) : \phi \in \mathrm{IBr}(B) \,\}$. Every $\phi \in \mathrm{IBr}(G)$ is contained in a unique p-block of G and $d_{\chi\phi}$ is zero whenever $\chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{IBr}(G)$ are contained in different p-blocks of G. Therefore, the decomposition matrix of G may be written in block diagonal form by means of the decomposition matrices of the p-blocks of G. For $\chi \in \mathrm{Irr}(G)$ the p-defect $d(\chi)$ is defined as the largest integer $d \geqslant 0$ such that $|G|/\chi(1)$

is divisible by p^d. To a p-block B of G we assign its *block constants* $k(B) := |\mathrm{Irr}(B)|$, $l(B) := |\mathrm{IBr}(B)|$, and $d(B) := \max\{\, d(\chi) : \chi \in \mathrm{Irr}(B)\,\}$, the *defect* of B. Moreover, an irreducible module that affords some $\varphi \in B$ is said to belong to B. We denote by 1_G the trivial character of G.

(1.2) Lemma. Let B be a p-block of G with bijections ι on $\mathrm{Irr}(B)$ and κ on $\mathrm{IBr}(B)$ such that $d_{\chi^\iota \phi^\kappa} = d_{\chi\phi}$ for all $\chi \in \mathrm{Irr}(B)$, $\phi \in \mathrm{IBr}(B)$. Define a class function $\rho \in \mathcal{C}(G, \mathbb{C})$ by

$$\rho(g) := \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\chi^\iota(g^{-1}).$$

Then $|\,\mathrm{Fix}_{\mathrm{Irr}(B)}(\iota)| = \langle 1_G, \rho\rangle$ and $|\,\mathrm{Fix}_{\mathrm{IBr}(B)}(\kappa)| = \langle 1_G|_{p'}, \rho|_{p'}\rangle = |\,\mathrm{Fix}_{\mathrm{IPr}(B)}(\iota)|$.

Proof. See [19, (2.4.2)]. The first assertion follows from

$$\langle 1_G, \rho\rangle = \sum_{\chi \in \mathrm{Irr}(B)} \sum_{g \in G} \chi(g^{-1})\chi^\iota(g) = \sum_{\chi \in \mathrm{Irr}(B)} \langle \chi^\iota, \chi\rangle = \sum_{\chi \in \mathrm{Irr}(B)} \delta_{\chi^\iota, \chi}.$$

As $\chi^\iota|_{p'} = \Delta(\chi^\iota) = \Delta(\chi)^\kappa = (\chi|_{p'})^\kappa$, now $\langle 1_G|_{p'}, \rho|_{p'}\rangle$ equals

$$\sum_{\chi \in \mathrm{Irr}(B)} \langle (\chi|_{p'})^\kappa, \chi|_{p'}\rangle = \sum_{\chi \in \mathrm{Irr}(B)} \sum_{\phi \in \mathrm{IBr}(B)} d_{\chi\phi} \langle \phi^\kappa, \chi|_{p'}\rangle = \sum_{\phi \in \mathrm{IBr}(B)} \langle \phi^\kappa, \Phi_\phi\rangle.$$

This is exactly the number of elements from $\mathrm{IBr}(B)$ that are invariant under κ. Extending ι on $\mathrm{Irr}(G)$ and κ on $\mathrm{IBr}(G)$ by acting as the identity on $\mathrm{Irr}(G)\backslash\mathrm{Irr}(B)$ and $\mathrm{IBr}(G)\backslash\mathrm{IBr}(B)$, respectively, by (1.1) this number is equal to $|\,\mathrm{Fix}_{\mathrm{IPr}(B)}(\iota)|$ as well. ∎

In the remainder of this section, let F be a subfield of $\mathcal{F}$ or f. Whenever we are speaking of a character corresponding to a module or a representation over F, then this means "ordinary character" if $F \subseteq \mathcal{F}$ and "Brauer character" otherwise.

If V is a vector space over F, every basis $(v_1, \ldots, v_n)$ of V induces a dual basis $(v_1^*, \ldots, v_n^*)$ of the dual vector space $V^* := \mathrm{Hom}_F(V, F)$ such that $v_j^{v_i^*} = \delta_{i,j}$ for all $i, j \in \{1, \ldots, n\}$. If V is a (right) FG-module, then V^* becomes a right FG-module denoted $V^\circledast$ by linear extension of $w^{(v^* g)} := (wg^{-1})^{v^*}$ for all $v^* \in V^*$, $w \in V$, $g \in G$. We call $V^\circledast$ the *dual* FG-module of V with corresponding *dual character* $\chi^\circledast$ of the character χ afforded by V. Denoting complex conjugation by $\circledast$ too, we have $\chi^\circledast(g) = \chi(g)^\circledast$ for all $g \in G$. (Recall that $\overline{\chi(g)}$ denotes the image of $\chi(g)$ under $\bar{\ } : \mathbb{Z}[\xi_{n,p}] \to f$.) Moreover, $\chi^\circledast$ is irreducible if and only if χ is, and we have $d_{\chi\phi} = d_{\chi^\circledast \phi^\circledast}$ for all $\chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{IBr}(G)$. Hence

$\chi \mapsto \chi^{\circledast}$ for $\chi \in \mathrm{Irr}(G)$ and $\phi \mapsto \phi^{\circledast}$ for $\phi \in \mathrm{IBr}(G)$ provides an example of maps ι and κ as in (1.1) and (1.2).

Another example is induced by the Frobenius automorphism $\wp : f \to f$, $x \mapsto x^p$. If δ is a matrix representation of G over f, then for $g \in G$ applying $\wp$ to every entry of g^δ again yields a matrix representation $\delta^\wp$ of G over f. Now $\wp$ lifts to a Galois automorphism $\hat{\wp}$ of $\mathbb{Q}(\xi_b)$ defined by $(\xi_b)^{\hat{\wp}} := \xi_b^p$ (as $p \nmid b$). The Brauer character $\phi^\wp$ of $\delta^\wp$ is given by $\phi^\wp(g) := \phi(g)^{\hat{\wp}}$ for $g \in G_{p'}$, where ϕ is the Brauer character of δ. Moreover, as ξ_{p^a} and ξ_b commute, $(\xi_m)^{\hat{\wp}} := \xi_{p^a}(\xi_b)^{\hat{\wp}}$ extends $\hat{\wp}$ to a Galois automorphism of $\mathbb{Q}(\xi_m)$. In particular, $\hat{\wp}$ acts on $\mathbb{Q}(\xi_{p^a})$ as the identity. Hence we may assign an *algebraically conjugate* representation $\varepsilon^{\hat{\wp}}$ to every $\mathcal{F}$-representation ε of G, acting "entrywise" by $\hat{\wp}$. If χ is the character of ε, then the character $\chi^{\hat{\wp}}$ of $\varepsilon^{\hat{\wp}}$ has values $\chi^{\hat{\wp}}(g) = \chi(g)^{\hat{\wp}}$ for $g \in G_{p'}$ and $\chi^{\hat{\wp}}(g) = \chi(g)$ for all p-elements $g \in G_p$. This means that $\hat{\wp}$ acts on $\mathrm{Irr}(G)|_{p'}$ just as $\wp$. It follows that $\phi \in \mathrm{IBr}(G)$ is a constituent of $\chi \in \mathrm{Irr}(G)|_{p'}$ if and only if $\phi^\wp \in \mathrm{IBr}(G)$ is a constituent of $\chi^{\hat{\wp}} \in \mathrm{Irr}(G)|_{p'}$. Thus $d_{\chi\phi} = d_{\chi^{\hat{\wp}}\phi^{\wp}}$ for all $\chi \in \mathrm{Irr}(G)$, $\phi \in \mathrm{IBr}(G)$.

Restriction and induction

Let H be a subgroup of G and F as above. If V is an FG-module that affords the character φ, then acting on V by H alone yields the *restricted* FH-module $V{\downarrow_H}$ with corresponding character $\varphi{\downarrow_H}$. Conversely, if W in an FH-module that affords the character ν, then the character $\nu{\uparrow^G}$ of the *induced* FG-module $W{\uparrow^G} := W \otimes_{FH} FG$ is given by

$$\nu{\uparrow^G}(g) = \sum_{i=1}^{n} \frac{|C_G(g)|}{|C_H(h_i)|} \nu(h_i)$$

for $g \in G$, where $h_1, \dots, h_n \in H$ are representatives of the H-conjugacy classes that fuse in g^G. Here it is to be understood that $\nu{\uparrow^G}(g) = 0$ if $g^G \cap H = \varnothing$.
Note that $V{\downarrow_H}$ is projective if V is projective and $W{\uparrow^G}$ is projective if W is projective. For an FH-module W and an FG-module V, the Frobenius–Nakayama reciprocity states

$$\mathrm{Hom}_{FG}(W{\uparrow^G}, V) \cong_F \mathrm{Hom}_{FH}(W, V{\downarrow_H}),$$
$$\mathrm{Hom}_{FG}(V, W{\uparrow^G}) \cong_F \mathrm{Hom}_{FH}(V{\downarrow_H}, W).$$

For arbitrary class functions $\nu \in \mathcal{C}(H, \mathbb{C})$ and $\varphi \in \mathcal{C}(G, \mathbb{C})$, we have $\langle \nu{\uparrow^G}, \varphi \rangle = \langle \nu, \varphi{\downarrow_H} \rangle$.

(1.3) Lemma. If V is an FG-module and W is an FH-module, then

$$(V{\downarrow_H} \otimes_F W){\uparrow^G} \cong V \otimes_F W{\uparrow^G} .$$

Proof. See [17, (6.10)]. Let T be a transversal of H in G. As

$$(V{\downarrow_H} \otimes_F W){\uparrow^G} = (V{\downarrow_H} \otimes_F W) \otimes_{FH} FG \cong V \otimes_F (W \otimes_{FH} FG) = V \otimes_F W{\uparrow^G},$$

we have an F-isomorphism α defined by $((v \otimes w) \otimes t)^\alpha = vt \otimes (w \otimes t)$ for $v \in V$, $w \in W$, $t \in T$. If $tg = ht'$ for $g \in G$, $h \in H$ and $t, t' \in T$, then

$$\begin{aligned}(((v \otimes w) \otimes t)g)^\alpha &= ((v \otimes w) \otimes ht')^\alpha = ((vh \otimes wh) \otimes t')^\alpha = vht' \otimes (wh \otimes t') \\ &= vtg \otimes (w \otimes tg) = vtg \otimes (w \otimes t)g = ((v \otimes w) \otimes t)^\alpha g.\end{aligned}$$

Thus α is an FG-isomorphism. ∎

Suppose that N is a normal subgroup of G. If W is an FN-module, then for $g \in G$ the g-*conjugate* $W^g \cong W \otimes g \subseteq W \otimes_{FN} FG$ of W is an FN-module with respect to $(w \otimes g)n = wn^{g^{-1}} \otimes g$ for $w \in W$, $n \in N$. Let $I_G(W) = \{ g \in G : W \otimes g \cong W \}$ denote the inertia subgroup of W in G.

(1.4) Theorem. Let V be an irreducible FG-module. If W is a composition factor of $V{\downarrow_N}$, then

$$V{\downarrow_N} \cong e \bigoplus_{i=1}^{r} W \otimes g_i$$

with $e \in \mathbb{N}$ and a transversal $\{g_1, \ldots, g_r\}$ of $I_G(W)$ in G. Suppose further that G/N is abelian and F is a splitting field for G with $\operatorname{char} F \nmid |G : N|$. Then

$$W{\uparrow^G} \cong e \bigoplus_{i=1}^{s} V \otimes_F U_i.$$

with pairwise non-isomorphic 1-dimensional FG-modules $U_1, \ldots, U_s$, each of which contains N in its kernel, and $n = |G : N| = e^2 rs$.

Proof. The first assertion is known by A. H. Clifford's theory, see [17, (6.2)] for instance. For the second one, see [63]. Set $n := |G : N|$. Since $F(G/N)$ is semisimple and commutative, it decomposes as a direct sum of 1-dimensional $F(G/N)$-modules $U_1, \ldots, U_n$, each of which becomes an FG-modules by inflation along N. Applying (1.3) with the trivial

FN-module yields

$$(V{\downarrow_N})\uparrow^G \cong V \otimes_F F(G/N) \cong \bigoplus_{j=1}^{n} V \otimes_F U_j.$$

As $(W \otimes g)\uparrow^G \cong W\uparrow^G$ for all $g \in G$, we have $(V{\downarrow_N})\uparrow^G = er \cdot W\uparrow^G$. Now $\mathcal{U} := \{U_1, \dots, U_n\}$ is an abelian group with respect to $\otimes_F$, so let $\{U_{i_1}, \dots, U_{i_s}\}$ be a transversal of the V-stabilizer $S := \{\, U_j \in \mathcal{U} : V \otimes_F U_j \cong V\}$ in $\mathcal{U}$. Then

$$er \cdot W\uparrow^G \cong \frac{n}{s} \bigoplus_{j=1}^{s} V \otimes_F U_{i_j}.$$

In particular, both $V{\downarrow_N}$ and $W\uparrow^G$ are completely reducible. By Frobenius–Nakayama reciprocity, the multiplicity $n/(ers)$ of V in $\operatorname{soc}(W\uparrow^G) \cong W\uparrow^G$ equals the multiplicity e of W in $\operatorname{hd}(V{\downarrow_N}) \cong V{\downarrow_N}$, thus $n = e^2 rs$. ∎

(1.5) Lemma ([24, Problem 6.3]). Assume that G/N is abelian and F is a splitting field for G with $\operatorname{char} F \nmid |G:N|$. If V is an irreducible FG-module that affords the character φ, and W is a composition factor of $V{\downarrow_N}$, the following are equivalent:

(1) $V{\downarrow_N} \cong e \cdot W$ and $e^2 = |G:N|$,

(2) $I_G(W) = G$ and φ vanishes on $G \setminus N$.

Proof. We apply (1.4). As $V{\downarrow_N}$ and $W\uparrow^G$ are completely reducible, we have

$$\dim_F \operatorname{Hom}_{FN}(W, V{\downarrow_N}) = \dim_F \operatorname{Hom}_{FG}(W\uparrow^G, V) = e.$$

In particular, if $V_N \cong e \cdot W$ such that $e^2 = |G:N|$, then $|G : I_G(W)| = 1$ and $W\uparrow^G = e \cdot V$. Let ν be the character afforded by W. As N is a union of G-conjugacy classes, the formula above immediately implies that ν^G vanishes on $G \setminus N$, and so does $\varphi = e^{-1}\nu\uparrow^G$. Conversely, suppose that $I_G(W) = G$. Then $V{\downarrow_N} \cong e \cdot W$ and $W\uparrow^G \cong e \bigoplus_{i=1}^{s} V \otimes U_i$ with pairwise non-isomorphic 1-dimensional FG-modules $U_1, \dots, U_s$. Denote by λ_i the character afforded by U_i. Then $N \leqslant \ker U_i$, hence $(\varphi \otimes \lambda_i){\downarrow_N} = \varphi{\downarrow_N}$. As φ vanishes outside of N, this means $\varphi \otimes \lambda_i = \varphi$ for all $i \in \{1, \dots, s\}$. Thus $s = 1$ and $e^2 = |G:N|$. ∎

The most important case for us is when N has index 2 in G and $\operatorname{char} F \neq 2$. Then there is exactly one non-trivial irreducible $F(G/N)$-module $\mathfrak{s}$, and inflating $\mathfrak{s}$ along N results in an irreducible 1-dimensional FG-module $\mathfrak{s}_G$, the "sign module" of G. We will denote its character as well as the corresponding representation by $\mathfrak{s}_G$, too. (We claim that the particular meaning will always be clear from the context.) In this way, every FG-module V has an *associate* FG-*module* $V^a := V \otimes \mathfrak{s}_G$, every representation δ of G has an *associate representation* $\delta^a := \delta \otimes \mathfrak{s}_G$, and every character φ of G has an *associate character* $\varphi^a := \varphi \otimes \mathfrak{s}_G$. We call φ *self-associate* if $\varphi = \varphi^a$.

Similarly, every FN-module W has a *conjugate* FN-*module* $W^c := W \otimes g$, where $g \in G$ such that $Ng \neq N$ (note that this is well-defined). The *conjugate representation* ε^c of a representation ε of N is given by $n^{\varepsilon^c} = (n^g)^\varepsilon$ for $n \in N$, and likewise the *conjugate character* ν^c of a character ν of N is defined. Also, we call ν *self-conjugate* if $\nu = \nu^c$.

(1.6) Corollary. Assume that $|G : N| = 2$. For $\mathcal{I} \in \{\mathrm{Irr}, \mathrm{IBr}\}$, let $\varphi \in \mathcal{I}(G)$ and $\nu \in \mathcal{I}(N)$. Then ν is a constituent of $\varphi{\downarrow_N}$ if and only if φ is a constituent of $\nu{\uparrow^G}$. In this case, suppose that $\operatorname{char} f \neq 2$ if $\mathcal{I} = \mathrm{IBr}$. Then, either

(a) $\varphi \neq \varphi^a, \quad \varphi{\downarrow_N} = \nu = \varphi^a{\downarrow_N}, \quad \nu{\uparrow^G} = \varphi + \varphi^a, \quad \nu = \nu^c, \quad$ or

(b) $\varphi = \varphi^a, \quad \varphi{\downarrow_N} = \nu + \nu^c, \quad \nu{\uparrow^G} = \varphi = \nu^c{\uparrow^G}, \quad \nu \neq \nu^c.$ ∎

Note that $\varphi \mapsto \varphi^a$ defines a bijection on both $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ such that $d_{\chi\phi} = d_{\chi^a\phi^a}$ for all $\chi \in \mathrm{Irr}(G)$, $\phi \in \mathrm{IBr}(G)$. Likewise, $\nu \mapsto \nu^c$ defines a bijection on both $\mathrm{Irr}(N)$ and $\mathrm{IBr}(N)$ such that $d_{\chi\nu} = d_{\chi^c\nu^c}$ for all $\chi \in \mathrm{Irr}(N)$, $\nu \in \mathrm{IBr}(N)$. By (1.1), we have $\Phi_{\varphi^a} = \Phi_\varphi^a$ for all $\varphi \in \mathrm{IBr}(G)$ and $\Phi_{\nu^c} = \Phi_\nu^c$ for all $\nu \in \mathrm{IBr}(N)$.

Suppose that $\varphi \in \mathrm{IBr}(G)$ with $\varphi{\downarrow_N} = \nu \in \mathrm{IBr}(N)$ and $\nu{\uparrow^G} = \varphi + \varphi^a$. Then $\varphi \neq \varphi^a$ and $\Phi_\varphi \neq \Phi_\varphi^a$. By the Frobenius–Nakayama reciprocity for class functions, we have

$$\langle \phi, \Phi_\nu{\uparrow^G} \rangle = \langle \phi{\downarrow_N}, \Phi_\nu \rangle = \begin{cases} 1 & \text{if } \phi \in \{\varphi, \varphi^a\}, \\ 0 & \text{if } \phi \notin \{\varphi, \varphi^a\}. \end{cases}$$

Thus $\Phi_{\nu\uparrow^G} = \Phi_\varphi + \Phi_\varphi^a = (\Phi_\nu){\uparrow^G}$. As $\langle \nu, \Phi_\varphi{\downarrow_N} \rangle = \langle \nu{\uparrow^G}, \Phi_\varphi \rangle = 1$ and $\langle \mu, \Phi_\varphi{\downarrow_N} \rangle = 0$ for all $\mu \in \mathrm{IBr}(N) \setminus \{\nu\}$, we get $\Phi_{\varphi\downarrow_N} = \Phi_\nu = (\Phi_\varphi){\downarrow_N}$.

Analogously, for $\nu \in \mathrm{IBr}(N)$ with $\nu{\uparrow^G} = \varphi \in \mathrm{IBr}(G)$ and $\varphi{\downarrow_N} = \nu + \nu^c$, we can deduce that $\Phi_{\nu\uparrow^G} = \Phi_\varphi = (\Phi_\nu){\uparrow^G}$ and $\Phi_{\varphi\downarrow_N} = \Phi_\nu + \Phi_\nu^c = (\Phi_\varphi){\downarrow_N}$.

(1.7) Corollary. Suppose $\operatorname{char} f \nmid 2 = |G : N|$. Let $\varphi \in \mathrm{IBr}(G)$ and $\nu \in \mathrm{IBr}(N)$ be as in (1.6).

(A) (1.6).(a) holds if and only if
$\Phi_\varphi \neq \Phi_\varphi^a, \quad \Phi_\varphi{\downarrow_N} = \Phi_\nu = \Phi_\varphi^a{\downarrow_N}, \quad \Phi_\nu{\uparrow^G} = \Phi_\varphi + \Phi_\varphi^a, \quad \Phi_\nu = \Phi_\nu^c.$

(B) (1.6).(b) holds if and only if
$\Phi_\varphi = \Phi_\varphi^a, \quad \Phi_\varphi{\downarrow_N} = \Phi_\nu + \Phi_\nu^c, \quad \Phi_\nu{\uparrow^G} = \Phi_\varphi = \Phi_\nu^c{\uparrow^G}, \quad \Phi_\nu \neq \Phi_\nu^c.$ ∎

Splitting fields

We finish this section quoting the Witt–Berman Theorem on the number of distinct irreducible representations of G over an arbitrary field E. As a consequence, we can derive the minimal size of a splitting field for G in characteristic p from the knowledge of the p-power map $g \mapsto g^p$ on $G_{p'}$.

Let E be an arbitrary field of characteristic $e \in \{0, p\}$ and let ϵ be a primitive dth root of unity over E, where $d = m$ is the exponent of G if $e = 0$, or $d = b$ is the p'-part of m if $e = p$. We interpret $G_{0'}$ as G and denote conjugacy of elements $x, y \in G$ by $x \sim_G y$. Let $I_d(E)$ be the set of integers $t \in \{1, \ldots, d-1\}$ coprime to e for which there exists an element σ in the Galois group of $E(\epsilon)$ over E such that $\epsilon^\sigma = \epsilon^t$. Then $x, y \in G$ are called E-conjugate if $x \sim_G y^t$ for some $t \in I_d(E)$. We write $x \sim_E y$ in this case. Note that this defines an equivalence relation on $G_{e'}$ and G-conjugacy implies obviously E-conjugacy. Hence we have a surjective map $\kappa : G_{e'}/\sim_G \to G_{e'}/\sim_E$, $[x]_{\sim_G} \mapsto [x]_{\sim_E}$. Note that κ is bijective if $E \in \{\mathcal{F}, f\}$, as $E(\epsilon) = E$ in that case. For the proof of the following theorem, see (21.5) and (21.25) in [13].

(1.8) Theorem (Witt–Berman). The number of isomorphism classes of irreducible EG-modules is equal to the number of E-conjugacy classes of $G_{e'}$. In particular, the map κ is bijective if and only if E is a splitting field for G. ∎

As an example, let us recover the well-known result that prime fields are splitting fields for $G = S_n$. Let E be $\mathbb{Q}$ or $GF(p)$. Recall that any two permutations $x, y \in S_n$ are conjugate in S_n if and only if the have the same cycle type (that is, the same number of cycles of the same length). Also, exponentiation of y by $t \in \mathbb{N}$ does not change the cycle type of y if t is coprime to each cycle length of y. Finally, recall that the order of an element of S_n is just the least common multiple of its cycle lengths. Since $t \in I_d(E)$ and d are coprime by definition, t is coprime to each possible cycle length of y, thus y and y^t are conjugate in S_n for all $t \in I_d(E)$. Hence $[x]_{\sim_E} = [y]_{\sim_E}$ if and only if $[x]_{\sim_G} = [y]_{\sim_G}$. Therefore, κ is bijective and E is a splitting field for S_n. In Section 5, we will derive the corresponding result for the double covers of S_n in characteristic p.

(1.9) Corollary. Let q be a power of p. Then $GF(q)$ is a splitting field for G if and only if $g \sim_G g^q$ for all $g \in G_{p'}$.

Proof. Denote $E := GF(q)$ and let ϵ be as above. Then the Galois group of $E(\epsilon)$ over E is cyclic of order k, say (which is equal to the degree of the minimal polynomial of ϵ

over E), and generated by the relative Frobenius automorphism $x \mapsto x^q$ of $E(\epsilon)$. Thus $I_b(E) = \{q, q^2, \ldots, q^k\}$. Suppose that $g \sim_G g^q$ for all $g \in G_{p'}$. It follows that $g \sim_G g^{q^i}$ for all $i \in \{1, \ldots, k\}$ and hence $g \sim_E h$ implies $g \sim_G h$ for $g, h \in G_{p'}$. Thus κ is bijective. Suppose conversely that E is a splitting field for G, and $g \in G_{p'}$. Note that $g \sim_E g^q$ since $g \sim_G g = (g^q)^{q^{k-1}}$. As κ is injective, $g \sim_G g^q$. ∎

2 Computing with characters

Suppose that we know $\mathrm{Irr}(G)$. We aim to compute $\mathrm{IBr}(G)$, or equivalently $\mathrm{IPr}(G)$, with respect to p dividing the order of G. We present some techniques that proved to be useful for computing (approximative) p-modular decomposition numbers of finite groups (for example [20, 28]), known as the MOC system [19]. See [19, 25, 27, 40, 41]. Throughout, let B be a union of p-blocks of G.

Basic sets

(2.1) Definition. A *basic set of Brauer characters* for B is a basis of $\langle \mathrm{IBr}(B)\rangle_{\mathbb{Z}}$ that is contained in $\langle \mathrm{IBr}(B)\rangle_{\mathbb{Z}^+}$. A *basic set of projective characters* for B is a basis of $\langle \mathrm{IPr}(B)\rangle_{\mathbb{Z}}$ that is contained in $\langle \mathrm{IPr}(B)\rangle_{\mathbb{Z}^+}$. Moreover, a basic set of Brauer characters for B that is contained in $\mathrm{Irr}(B)|_{p'}$ is called a *special basic set* for B.

If the context is clear, we may speak of a basic set for short rather than a basic set of Brauer characters or a basic set of projective characters. Notationally, a '$\mathcal{B}$' indicates Brauer characters whereas a '$\mathcal{P}$' indicates projective characters. In this way, let $\mathcal{B}$ be a $\mathbb{Z}$-linearly independent set of Brauer characters of G. If $\mathcal{B}$ is a basic set for B, clearly every $\chi \in \mathrm{Irr}(G)|_{p'}$ decomposes with integer coefficients over $\mathcal{B}$. Conversely, suppose that $\mathrm{Irr}(B)|_{p'}$ is contained in $\langle \mathcal{B}\rangle_{\mathbb{Z}}$. Since $\Delta|_B$ is surjective (see [24, (15.28)]), for every $\phi \in \mathrm{IBr}(B)$ there is some $\chi \in \langle \mathrm{Irr}(B)\rangle_{\mathbb{Z}}$ such that $\phi = \Delta(\chi) = \chi_{p'}$. Thus $\mathrm{IBr}(B) \subseteq \langle \mathcal{B}\rangle_{\mathbb{Z}}$ and $\mathcal{B}$ is a basic set for B.

(2.2) Proposition ([19, (3.1.2)]). A set $\mathcal{B}$ of Brauer characters is a basic set for B if and only if $\mathcal{B}$ is linearly independent over $\mathbb{Z}$ and $\mathrm{Irr}(B)|_{p'} \subseteq \langle \mathcal{B}\rangle_{\mathbb{Z}}$. In particular, $\mathcal{B} \subseteq \mathrm{Irr}(B)|_{p'}$ is a special basic set for B if and only if $\mathcal{B}$ is $\mathbb{Z}$-linearly independent and $\mathrm{Irr}(B)|_{p'} \subseteq \langle \mathcal{B}\rangle_{\mathbb{Z}}$. ∎

For our purposes, we usually begin with computing a special basic set[#1] for G by

[1] It is an open question whether special basic sets exist in general. Positive answers have been given for sporadic simple groups [19], p-solvable groups (Fong–Swan), groups (blocks) of cyclic defect (Brauer–Dade), and some groups of Lie type in non-defining or defining characteristic (see [5] and the references therein). For the symmetric group S_n a special basic set indexed by all p-regular partitions of n is known by James, see (5.3.1). Recently, a special basic set for S_n that restricts to one for A_n was found [6, 7].

finding a maximal $\mathbb{Z}[\xi_b]$-linearly independent subset $\mathcal{B}(G) \subseteq \mathrm{Irr}(G)|_{p'}$, swapping suitable elements from $\mathcal{B}(G)$ and $(\mathrm{Irr}(G)|_{p'}) \setminus \mathcal{B}(G)$ unless $\mathrm{Irr}(G)|_{p'}$ decomposes integrally over $\mathcal{B}(G)$. Then $\mathcal{B} := \mathcal{B}(G) \cap \mathrm{Irr}(B)|_{p'}$ is a special basic set for B. In particular, the number $\ell := |\mathcal{B}| = |\mathrm{IBr}(B)|$ is just the $\mathbb{Z}[\xi_b]$-rank of $\mathrm{Irr}(B)|_{p'}$. Note that $\mathrm{Irr}(B)$ can be computed by means of central characters from the ordinary character table of G alone, see Section 1.

A Brauer character $\phi = \sum_{\chi \in \mathcal{B}(G)} m_\chi \chi$ of G can be stored as the list $(m_\chi : \chi \in \mathcal{B}(G))$ of integer coefficients with respect to $\mathcal{B}(G)$. Then the projection $\phi|_B$ of ϕ onto B is simply given by $m := (m_\chi : \chi \in \mathcal{B})$ because $\mathcal{B} = \mathcal{B}(G) \cap \mathrm{Irr}(B)|_{p'}$. Moreover, suppose that $\Phi = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \chi$ is a projective character. Then $\langle \chi, \Phi \rangle = n_\chi$ for all $\chi \in \mathrm{Irr}(G)$ and Φ may be projected onto B as $\Phi|_B = \sum_{\chi \in \mathrm{Irr}(B)} n_\chi \chi$. In fact, Φ can be recovered from $n := (n_\chi : \chi \in \mathcal{B})$ and knowing how the elements of $(\mathrm{Irr}(B)|_{p'}) \setminus \mathcal{B}(G)$ decompose over $\mathcal{B}$. In particular,

$$\langle \phi|_B, \Phi|_B \rangle = \sum_{\chi \in \mathcal{B}} m_\chi \langle \chi, \Phi \rangle = \sum_{\chi \in \mathcal{B}} m_\chi n_\chi = m \cdot n^t.$$

(2.3) Lemma. If $\Phi = \sum_{\chi \in \mathrm{Irr}(B)} n_\chi \chi$ is a projective character of G and $n \in \mathbb{N}$ such that $n \mid n_\chi$ for all $\chi \in \mathrm{Irr}(B)$, then Φ/n is a projective character of G.

Proof. See [27, (1.23)]. Let $\Phi = \sum_{\phi \in \mathrm{IBr}(B)} m_\phi \Phi_\phi$ be the decomposition of Φ in terms of the projective indecomposable characters of B. By the surjectivity of $\Delta|_B$, each element $\phi \in \mathrm{IBr}(B)$ may be written as $\phi = \sum_{\chi \in \mathrm{Irr}(B)} z_{\phi\chi} \chi|_{p'}$ with suitable coefficients $z_{\phi\chi} \in \mathbb{Z}$ for $\chi \in \mathrm{Irr}(B)$. Since

$$m_\phi = \langle \phi, \Phi \rangle = \sum_{\chi \in \mathrm{Irr}(B)} z_{\phi\chi} \langle \chi, \Phi \rangle = \sum_{\chi \in \mathrm{Irr}(B)} z_{\phi\chi} n_\chi$$

implies $n \mid m_\phi$ for all $\phi \in \mathrm{IBr}(B)$, clearly Φ/n is projective. ∎

A class function $\varphi \in \mathcal{C}(G_{p'}, \mathbb{C})$ may be written as an l-tuple $\varphi = (\varphi(g_1), \ldots, \varphi(g_l))$ with respect to the representatives $g_1, \ldots, g_l$ of the p-regular classes of G. For a k-tuple $M = (\varphi_1, \ldots, \varphi_k)$ of class functions on $G_{p'}$, let $[M]$ denote the $k \times l$-matrix whose ith row is φ_i for $i \in \{1, \ldots, k\}$. Moreover, for $M, N \in \mathcal{C}(G_{p'}, \mathbb{C})^k$ we define the *matrix of mutual scalar products* $U(M, N) := [\langle \mu, \nu \rangle]_{\mu \in M, \nu \in N}$. (In due course we often consider sets M, N rather than tuples, assuming a given ordering. Later on, we will label the rows and columns of $U(M, N)$ anyway.) Note that $U(M, N) = [M]C[N^{\circledast}]^t$, where C is defined as $\mathrm{diag}(|C_G(g_1)|^{-1}, \ldots, |C_G(g_l)|^{-1})$. Subject to a suitable ordering, the duality of $\mathrm{IBr}(G)$ and $\mathrm{IPr}(G)$ is expressed as

$$\mathcal{U}(\mathrm{IBr}(G), \mathrm{IPr}(G)) = [\mathrm{IBr}(G)]C[\mathrm{IPr}(G)^{\circledast}]^t = I_l.$$

(2.4) Lemma. Let $\mathcal{B}$ and $\mathcal{P}$ be ℓ-tuples of Brauer characters and projective characters of G, respectively. Then $\mathcal{B}$ and $\mathcal{P}$ are basic sets for B if and only if their matrix of mutual scalar products is invertible over $\mathbb{Z}$.

Proof. See [19, (3.1.4)]. As we may write $[\mathcal{B}] = U_1[\mathrm{IBr}(B)]$ and $[\mathcal{P}] = U_2^t[\mathrm{IPr}(B)]$ with suitable matrices U_1 and U_2 over $\mathbb{Z}^+$, the claim follows from

$$\mathcal{U}(\mathcal{B}, \mathcal{P}) = U_1[\mathrm{IBr}(B)]C[\mathrm{IPr}(B)^{\circledast}]^t U_2 = U_1 U_2.$$

∎

(2.5) Corollary. Let $\mathcal{BS}$ and $\mathcal{PS}$ be basic sets for B. Then $\mathcal{BS} = \mathrm{IBr}(B)$ and $\mathcal{PS} = \mathrm{IPr}(B)$ if and only if $\mathcal{U}(\mathcal{BS}, \mathcal{PS})$ is a permutation matrix. ∎

Atoms

Let $\mathcal{BS}$ be a basic set of Brauer characters for B. Since $\mathrm{IBr}(B)$ is linearly independent over $\mathbb{C}$ and contained in $\langle \mathcal{BS} \rangle_{\mathbb{C}}$, we deduce from

$$\ell = l(B) = \dim \langle \mathrm{IBr}(B) \rangle_{\mathbb{C}} \leqslant \dim \langle \mathcal{BS} \rangle_{\mathbb{C}} \leqslant \mathrm{rk}\, \langle \mathcal{BS} \rangle_{\mathbb{Z}} = \ell$$

that indeed $\mathcal{BS}$ is a basis of $\langle \mathrm{IBr}(B) \rangle_{\mathbb{C}}$. Its dual basis $\mathcal{BS}^*$ with respect to $\langle \cdot, \cdot \rangle$ is a basis of $\langle \mathrm{IBr}(B) \rangle_{\mathbb{C}}^* = \langle \mathrm{IPr}(B) \rangle_{\mathbb{C}}$ with $\langle \mathcal{BS}^* \rangle_{\mathbb{Z}} = \langle \mathrm{IPr}(B) \rangle_{\mathbb{Z}}$. Note that the elements of $\mathcal{BS}^*$ need not be proper projective characters. Nevertheless, the following shows that every projective character can be decomposed over $\mathcal{BS}^*$ with non-negative integral coefficients, and the analogue statement for Brauer characters holds too.

(2.6) Definition. Let $\mathcal{BS} = (\phi_1, \ldots, \phi_\ell)$ be a basic set of Brauer characters for B. The elements of the dual basis $(\phi_1^*, \ldots, \phi_\ell^*)$ of $\mathcal{BS}$ (with respect to $\langle \cdot, \cdot \rangle$) are called *projective atoms* corresponding to $\mathcal{BS}$. The elements of the dual basis $(\Phi_1^*, \ldots, \Phi_\ell^*)$ of a basic set of projective characters $\mathcal{PS} = (\Phi_1, \ldots, \Phi_\ell)$ are called *Brauer atoms* corresponding to $\mathcal{PS}$.

(2.7) Lemma. If $\mathcal{BS}$ is a basic set for B with corresponding projective atoms $\mathcal{PA}$, then $\langle \mathrm{IPr}(B) \rangle_{\mathbb{Z}^+} \subseteq \langle \mathcal{PA} \rangle_{\mathbb{Z}^+}$. Analogously, if $\mathcal{PS}$ is a basic set for B with corresponding Brauer atoms $\mathcal{BA}$, then $\langle \mathrm{IBr}(B) \rangle_{\mathbb{Z}^+} \subseteq \langle \mathcal{BA} \rangle_{\mathbb{Z}^+}$.

Proof. Cf. [19, (3.2.1), (3.2.2)]. We may write $\mathrm{IBr}(B) = U_1^{-1}[\mathcal{BS}]$ and $\mathrm{IPr}(B) = U_2^t[\mathcal{PA}]$ with suitable invertible matrices U_1 over $\mathbb{Z}^+$ and U_2 over $\mathbb{C}$. Now $[\mathcal{BS}]C[\mathcal{PA}^{\circledast}]^t = I_\ell$ and

$$U_1^{-1}U_2^{\circledast} = U_1^{-1}[\mathcal{BS}]C[\mathcal{PA}^{\circledast}]^t U_2^{\circledast} = [\mathrm{IBr}(B)]C[\mathrm{IPr}(B)^{\circledast}]^t = I_\ell.$$

Thus $U_2^{\circledast} = U_1 = U_2$ has only non-negative integer entries. ∎

Let $\mathcal{BS} = (\phi_1, \ldots, \phi_\ell)$ and $\mathcal{PS} = (\Phi_1, \ldots, \Phi_\ell)$ be basic sets for B with corresponding projective atoms $\mathcal{PA} = (\phi_1^*, \ldots, \phi_\ell^*)$ and Brauer atoms $\mathcal{BA} = (\Phi_1^*, \ldots, \Phi_\ell^*)$. Suppose that $\phi_i = \sum_{j=1}^\ell m_{ij}\Phi_j^*$ and $\Phi_i = \sum_{j=1}^\ell n_{ij}\phi_j^*$. Then $\langle \phi_i, \Phi_j\rangle = m_{ij}$ and $\langle \phi_j, \Phi_i\rangle = n_{ij}$ for all $j \in \{1, \ldots, \ell\}$. Hence $[\mathcal{BS}] = U[\mathcal{BA}]$ and $[\mathcal{PS}] = U^t[\mathcal{PA}]$ with $U := U(\mathcal{BS}, \mathcal{PS})$. This means the ith row of U describes the decomposition of $\phi_i \in \mathcal{BS}$ in terms of $\mathcal{BA}$, and the ith column of U describes the decomposition of $\Phi_i \in \mathcal{PS}$ in terms of $\mathcal{PA}$. In particular, $[\mathcal{BA}] = U^{-1}[\mathcal{BS}]$ and $[\mathcal{PA}] = (U^{-1})^t[\mathcal{PS}]$.
The following result says that we have equality in (2.7) if and only if $\mathcal{BS} = \mathrm{IBr}(B)$ respectively $\mathcal{PS} = \mathrm{IPr}(B)$.

(2.8) Lemma. If $\mathcal{BS}$ and $\mathcal{PS}$ are basic sets for B with corresponding atoms $\mathcal{PA}$ and $\mathcal{BA}$, respectively, then $\langle \mathrm{IBr}(B)\rangle_{\mathbb{Z}^+} \cap \mathcal{BA} \subseteq \mathrm{IBr}(B)$ and $\langle \mathrm{IPr}(B)\rangle_{\mathbb{Z}^+} \cap \mathcal{PA} \subseteq \mathrm{IPr}(B)$. In particular, $\mathcal{BS} \cap \mathcal{BA} \subseteq \mathrm{IBr}(B)$ and $\mathcal{PS} \cap \mathcal{PA} \subseteq \mathrm{IPr}(B)$.

Proof. See [19, (3.2.3)]. Let $\mathcal{BS}$ and $\mathcal{BA}$ be as above and $\mathrm{IBr}(B) = (\underline{\phi}_1, \ldots, \underline{\phi}_\ell)$. If Φ_k^* is a Brauer character, then we may write $\Phi_k^* = \sum_{i=1}^\ell m_i\underline{\phi}_i$ with $m_i \in \mathbb{Z}^+$. Also, by (2.7), each $\underline{\phi}_i$ may be decomposed in terms of $\mathcal{BA}$ as $\underline{\phi}_i = \sum_{j=1}^\ell z_{ij}\Phi_j^*$ with $z_{ij} \in \mathbb{Z}^+$. It follows that

$$\Phi_k^* = \sum_{i=1}^{\ell} m_i\underline{\phi}_i = \sum_{j=1}^{\ell}\sum_{i=1}^{\ell} m_i z_{ij}\Phi_j^*.$$

Hence there is an index i_0 such that $m_{i_0}z_{i_0k} = 1$ (that is, $m_{i_0} = z_{i_0k} = 1$) and $m_iz_{ij} = 0$ for all $i, j \in \{1, \ldots, \ell\}$, $i \neq i_0$. As for each i there is at least one j such that $z_{ij} \neq 0$, we have $\Phi_k^* = \underline{\phi}_{i_0}$. The proof of the second claim is completely analogous. ∎

Proving minimality

Let $\mathcal{BS}$, $\mathcal{PS}$ and their atoms $\mathcal{PA}$, $\mathcal{BA}$ be as above. Testing Brauer characters for irreducibility and projective characters for indecomposability can be done as follows.

(2.9) Definition. A *part* of a Brauer character $\phi = \sum_{i=1}^{\ell} n_i \Phi_i^* \in \langle \mathrm{IBr}(B) \rangle_{\mathbb{Z}^+}$ is a subsum

$$\sum_{i=1}^{\ell} m_i \Phi_i^* \quad \text{such that} \quad 0 \leqslant m_i \leqslant n_i \quad \text{for all} \quad i \in \{1, \ldots, \ell\}.$$

A *part* of a projective character $\Phi = \sum_{i=1}^{\ell} n_i \phi_i^* \in \langle \mathrm{IPr}(B) \rangle_{\mathbb{Z}^+}$ is defined analogously.

It follows from (2.7) that the irreducible constituents of a Brauer character ϕ are among its parts. Then ϕ must be irreducible if no part of ϕ is a proper Brauer character.

(2.10) Lemma ([19, (5.2.2)]). If $\phi \in \langle \mathrm{IBr}(B) \rangle_{\mathbb{Z}^+}$ such that for every part $\phi' \notin \{0, \phi\}$ of ϕ there is a projective character Φ with $\langle \phi', \Phi \rangle < 0$, then $\phi \in \mathrm{IBr}(B)$. Analogously, if $\Phi \in \langle \mathrm{IPr}(B) \rangle_{\mathbb{Z}^+}$ such that for every part $\Phi' \notin \{0, \Phi\}$ of Φ there is a Brauer character ϕ with $\langle \phi, \Phi' \rangle < 0$, then $\Phi \in \mathrm{IPr}(B)$. ∎

Since ϕ' is a part of ϕ if and only if $\phi - \phi'$ is one, we note that for every non-trivial part ϕ' of ϕ the run through all projective characters Φ at hand may be shortened by testing whether $\langle \phi', \Phi \rangle < 0$ or $\langle \phi - \phi', \Phi \rangle < 0$.

Embarking from basic sets $\mathcal{BS}$ and $\mathcal{PS}$ for B we aim at improving them gradually until either one, $\mathrm{IBr}(B)$ or $\mathrm{IPr}(B)$, is found or at least approximated closely. In this work, improvements are mainly deduced from *relations*, that is, decompositions of Brauer characters in terms of $\mathcal{BS}$ or projective characters in terms of $\mathcal{PS}$. There is a more involved method of improving basic sets than improving "by hand". For this procedure (and the test of (2.10) too) methods of linear programming can be used. However, as it will not be needed in the proofs of our results, we refer the reader to [19, 27].

Finding basic sets

We complete this section by explaining a technique for finding basic sets. For instance, if we are given a set $\mathcal{C}$ of Brauer characters, we need to find a basis $\mathcal{B}$ of $\langle \mathcal{C} \rangle_{\mathbb{Z}}$ that consists only of proper Brauer characters. More generally, suppose that L is a finite dimensional $\mathbb{Q}$-vector space with basis $\mathcal{A} := \{\alpha_1, \ldots, \alpha_n\}$ and let $\mathcal{L} := \langle \mathcal{A} \rangle_{\mathbb{Z}}$ be the corresponding $\mathbb{Z}$-lattice. We define $\mathcal{L}^+ := \langle \mathcal{A} \rangle_{\mathbb{Z}^+}$ and call subsets of $\mathcal{L}^+$ *positive*. In this setting, the problem reads as follows: given a finite positive set $\mathcal{C} \subset \mathcal{L}^+$, find a positive basis of $\langle \mathcal{C} \rangle_{\mathbb{Z}}$. A solution to this problem is given by R. Parker's `FBA` algorithm [19, (5.1.3)]. We present a variant proposed to the author by J. Müller. The basic outline is as follows.

(2.11) Algorithm. FBA

Input: a finite positive subset $\mathcal{C}$ of $\mathcal{L}$

Output: a positive basis $\mathcal{B}$ of $\langle \mathcal{C} \rangle_{\mathbb{Z}}$

```
B ← {}
for γ ∈ C do
  if γ ∉ ⟨B⟩_Z then
    if γ ∉ ⟨B⟩_Q then
      B ← B ∪ {γ}
    else
      B ← changebasis(B, γ)     #change B such that γ ∈ ⟨B⟩_Z but still B ⊂ L^+
    end if
  end if
end for
return B
```

The crucial point in (2.11) is the procedure changebasis: given an element γ and a positive, $\mathbb{Q}$-linearly independent set $\mathcal{B}$ such that γ is contained in the $\mathbb{Q}$-span of $\mathcal{B}$, its assignment is to construct a positive, linearly independent set $\mathcal{B}'$ such that γ can be decomposed integrally over $\mathcal{B}'$. Breaking this task into smaller pieces, changebasis uses a certain reduction that we discuss first.

(2.12) Procedure. reducepair

Input: $\mathcal{B} = \{\beta_1, \ldots, \beta_j\} \subset \mathcal{L}^+$, $\gamma \in \mathcal{L}$, $c = (c_1, \ldots, c_j) \in \mathbb{Z}^j$, $0 < b \in \mathbb{Z}$ such that $\beta_1, \ldots, \beta_j$ are linearly independent over $\mathbb{Z}$ and $b\gamma = \sum_{i=1}^{j} c_i \beta_i$

Output: $\beta_j' \in \mathcal{L}^+$, $\gamma' \in \mathcal{L}$, $0 < b' \in \mathbb{Z}$ such that $B' := \{\beta_1, \ldots, \beta_{j-1}, \beta_j'\}$ is linearly independent over $\mathbb{Z}$, $\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}', \gamma' \rangle_{\mathbb{Z}}$, and $b'\gamma' = \sum_{i=1}^{j-1} c_i \beta_i$ (see (2.13))

```
if c_j = 0 then
  (β'_j, γ', b') ← (β_j, γ, b)
else
  d ← gcd(b, c_j)     #the positive greatest common divisor of b and c_j
  if d = b then
```

```
    r ← 1, s ← 0
  else
    find suitable r, s ∈ ℤ such that d = rb + sc_j
  end if
  for i = 1, ..., j − 1 do
    e_i ← min{ e ∈ ℤ : e + sc_i/b ⩾ 0 }
  end for
  β'_j ← Σ_{i=1}^{j−1} e_i β_i + rβ_j + sγ    #if d = b, then β'_j = β_j
  γ' ← (1/d)(bγ − c_j β_j)
  b' ← d
end if
return (β'_j, γ', b')
```

(2.13) Lemma. In the setting of (2.12), we have $\beta_j' \in \mathcal{L}^+$, $\gamma' \in \mathcal{L}$ and $0 < b' \in \mathbb{Z}$ such that $\mathcal{B}'$ is linearly independent over $\mathbb{Z}$, $\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}', \gamma' \rangle_{\mathbb{Z}}$, and $b'\gamma' = \sum_{i=1}^{j-1} c_i \beta_i$.

Proof. Assume that $c_j \neq 0$, as otherwise the claims hold trivially. By definition, we have $0 < b' = d = \gcd(b, c_j) \in \mathbb{Z}$ and $\gamma' \in \langle \gamma, \beta_j \rangle_{\mathbb{Z}} \subseteq \mathcal{L}$ with $b'\gamma' = \sum_{i=1}^{j-1} c_i \beta_i$; moreover, $\beta_j' = \sum_{i=1}^{j-1} e_i \beta_i + r\beta_j + s\gamma \in \mathcal{L}$. In particular,

$$\langle \mathcal{B}', \gamma' \rangle_{\mathbb{Z}} = \langle \beta_1, \ldots, \beta_{j-1}, r\beta_j + s\gamma, -(c_j/d)\beta_j + (b/d)\gamma \rangle_{\mathbb{Z}} \subseteq \langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}}.$$

As

$$\begin{bmatrix} r & s \\ -c_j/d & b/d \end{bmatrix}$$

is invertible over $\mathbb{Z}$, we have

$$\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}', \gamma' \rangle_{\mathbb{Z}}.$$

It remains to verify $\beta_j' \in \mathcal{L}^+$. Recall that $\mathcal{L} = \langle \mathcal{A} \rangle_{\mathbb{Z}}$ where $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Q}$-basis of L; as

$$\beta_j' = \sum_{i=1}^{j-1} \left(e_i + s\frac{c_i}{b} \right) \beta_i + \left(r + s\frac{c_j}{b} \right) \beta_j$$

with

$$e_i + s\frac{c_i}{b} \geqslant 0 \quad \text{for all } i < j$$

and

$$r + s\frac{c_j}{b} = \frac{d}{b} > 0,$$

we see that β_j' is contained in the $\mathbb{Q}^+$-span of $\beta_1, \ldots, \beta_j$. Thus

$$\beta_j' = \sum_{k=1}^{n} a_k \alpha_k = \sum_{i=1}^{j} \frac{p_i}{q_i} \beta_i$$

for suitable $a_k \in \mathbb{Z}$ and $p_i/q_i \in \mathbb{Q}^+$. Furthermore, as $\{\beta_1, \ldots, \beta_j\}$ is positive, there exist $b_{ik} \in \mathbb{Z}^+$ such that $\beta_i = \sum_{k=1}^{n} b_{ik}\alpha_k$ for all i. Since $\mathcal{A}$ is $\mathbb{Q}$-linearly independent, we can deduce that

$$a_k = \sum_{i=1}^{j} \frac{p_i}{q_i} b_{ik} \in \mathbb{Z}^+$$

and therefore $\beta_j' \in \mathcal{L}^+$. Finally, if there exist $z_1, \ldots, z_j \in \mathbb{Z}$ such that $z_j \neq 0$ and

$$\sum_{i=1}^{j-1} z_i \beta_i + z_j \beta_j' = \sum_{i=1}^{j-1} \left(z_i + z_j \frac{p_i}{q_i} \right) \beta_i + z_j \frac{d}{b} \beta_j = 0,$$

then $z_j d = 0$, a contradiction. ∎

Now we can introduce the procedure `changebasis` used in (2.11).

(2.14) Procedure. `changebasis`

`Input:` a $\mathbb{Z}$-linearly independent set $\mathcal{B} = \{\beta_1, \ldots \beta_k\} \subset \mathcal{L}^+$ and $\gamma = \sum_{i=1}^{k} a_i/b_i \beta_i \in \mathcal{L}$ with $a_i, b_i \in \mathbb{Z}$, $b_i > 0$ such that $\gcd(a_i, b_i) = 1$ $(i \in \{1, \ldots, k\})$ and $\gamma \notin \langle \mathcal{B} \rangle_{\mathbb{Z}}$

`Output:` a $\mathbb{Z}$-linearly independent set $\mathcal{B}' \subset \mathcal{L}^+$ such that $\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}' \rangle_{\mathbb{Z}}$ (see (2.15))

$j \leftarrow k$
$b \leftarrow \mathrm{lcm}(b_1, \ldots, b_j)$ #the positive least common multiple of $b_1, \ldots, b_j$
$c \leftarrow b \cdot (a_1/b_1, \ldots, a_j/b_j)$
$(\beta_j', \gamma', b') \leftarrow$ `reducepair`$(\mathcal{B}, \gamma, c, b)$
`while` $b' > 1$ `do`
  $j \leftarrow j - 1$
  $\mathcal{B} \leftarrow \{\beta_1, \ldots, \beta_j\}$
  $c \leftarrow (c_1, \ldots, c_j)$
  $b \leftarrow b'$

```
    (β'_j, γ', b') ← reducepair(B, γ', c, b)    # b' = 1 for some j ≥ 1, see (2.15)
  end while
  B' ← {β_1, ..., β_{j−1}, β'_j, ... β'_k}
  return B'
```

(2.15) Lemma. In the setting of (2.14), we have $b' = 1$ for some $j \in \{1, \ldots, k\}$. Moreover, $\mathcal{B}'$ is positive, linearly independent over $\mathbb{Z}$, and $\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}' \rangle_{\mathbb{Z}}$.

Proof. In order to prove that eventually $b' = 1$, define integers D_j and B_j recursively by $B_k := \operatorname{lcm}(b_1, \ldots, b_k) > 0$, and $B_{j-1} := D_j := \gcd(B_j, c_j) > 0$ for $j = k, k-1, \ldots$; then the input value b of `reducepair` during the while-loop iteration depending on j is B_j, and the corresponding output value b' is D_j.
We claim there exists an index j such that $D_j = 1$. Note the consecutive divisibility

$$D_j \mid B_j = D_{j+1} \mid B_{j+1} \quad \ldots \quad D_{k-1} \mid B_{k-1} = D_k \mid B_k$$

given by definition for every $j \geqslant 1$, and assume that $D_1 > 1$. Let $i \in \{1, \ldots, k\}$. We have $D_1 \mid D_i \mid c_i = B_k a_i / b_i$ and therefore $b_i \mid (B_k a_i / D_1)$. Now $\gcd(a_i, b_i) = 1$ implies that $b_i \mid (B_k / D_1) < B_k$ for all i, contradicting our choice of B_k. Thus $b' = D_j = 1$ for some $j \geqslant 1$. In that case, note that $\gamma' = \sum_{i=1}^{j-1} c_i \beta_i \in \langle \beta_1 \ldots, \beta_{j-1} \rangle_{\mathbb{Z}}$. Thus, applying (2.13) successively, we have

$$\langle \beta_1, \ldots, \beta_k, \gamma \rangle_{\mathbb{Z}} = \langle \beta_1, \ldots, \beta_{j-1}, \beta'_j, \ldots \beta'_k, \gamma' \rangle_{\mathbb{Z}} = \langle \mathcal{B}' \rangle_{\mathbb{Z}}.$$

Also $\beta_1, \ldots, \beta_k \in \langle \mathcal{B}' \rangle_{\mathbb{Q}}$ implies that $\mathcal{B}'$ is linearly independent over $\mathbb{Q}$, and thus over $\mathbb{Z}$. Finally, it follows from (2.13) that $\mathcal{B}'$ is positive. ∎

Suppose that we are in the setting of (2.11) with $\gamma = \gamma_l \in \mathcal{C} = \{\gamma_1, \ldots, \gamma_m\}$ and a positive, linearly independent set $\mathcal{B}$ such that $\langle \gamma_1, \ldots, \gamma_{l-1} \rangle_{\mathbb{Z}} = \langle \mathcal{B} \rangle_{\mathbb{Z}}$ and γ is contained in the $\mathbb{Q}$-span but not in the $\mathbb{Z}$-span of $\mathcal{B}$. By (2.15), applying (2.14) creates a positive and linearly independent set $\mathcal{B}'$ such that $\langle \mathcal{B}, \gamma \rangle_{\mathbb{Z}} = \langle \mathcal{B}' \rangle_{\mathbb{Z}}$, and therefore $\langle \gamma_1, \ldots, \gamma_l \rangle_{\mathbb{Z}} = \langle \mathcal{B}' \rangle_{\mathbb{Z}}$. Thus we have proved

(2.16) Theorem (cf. [19, (5.1.4)]). Algorithm (2.11) terminates and its output is a basis of $\langle \mathcal{C} \rangle_{\mathbb{Z}}$ which is contained in $\mathcal{L}^+$. ∎

In the practical use of (2.11), linear equations $x \cdot \mathcal{B} = \gamma$ must be handled. For finding integral solutions x, a suitable method based on p-adic approximation can be used, see for instance [14, 40].#2 Combining this technique with ordinary Gaussian elimination for rational solutions x, it is convenient to store additionally a semi-echelon form B of $\mathcal{B}$ with coefficients C such that $\mathcal{B} = CB$, where all changes to $\mathcal{B}$ are applied simultaneously to C.

[2] In GAP , see `Decomposition()` [18, ref. manual 25.4].

3 Computing with modules

Characters alone usually do not provide enough information to fully determine the p-modular decomposition matrix of G. Nevertheless, additional information can often be gained from explicitly analyzing suitable representations of G over some (preferably small) subfield F of f. We use the C–MeatAxe programs [62] based originally on [61]. Various contributions, for instance [39, 43], are among the C–MeatAxe features. These are complemented by our own programs written in GAP. The GAP package AtlasRep [70] can be used as a link between both systems. A more thorough description of the MeatAxe programs and underlying concepts can be found in [38, 39].

Let A be a finite dimensional unital algebra over a finite field F. For example, the group algebra $A = FG$ of G over $F \leqslant f$ as in Section 1. Let $\delta : A \to \mathrm{End}_F(V)$, $a \mapsto a_V$ be a representation of A on a vector space V over F of finite dimension d. Equally well, we may consider V as an A-module. Moreover, let $\mathcal{A}$ be a list of generators for A. Then δ is completely determined by the images $\mathcal{A}_V := \mathcal{A}^\delta = (a_V : a \in \mathcal{A})$. Choosing a basis $\mathcal{B}$ of V, δ can be realized as a matrix representation of A on F^d. If the latter is important, we write ${}_{\mathcal{B}}[a]_{\mathcal{B}}$ instead of a_V. Vice versa, every such matrix representation turns F^d into an A-module that is isomorphic to V and affords a representation equivalent to δ. For all practical applications, it should be noted that none of the following methods are of much use unless the degree d and the realization of F allow feasible input for the computer's hard- and software in use.#1

Down to the bone: the MeatAxe

The reader may think of a toolbox of programs for the working representation theorist. One of the main tools, the program chop determines the irreducible constituents of δ from input $\mathcal{A}_V \subseteq F^{d\times d}$. It computes a composition series of the underlying A-module V and returns irreducible matrix representations of A on the composition factors of V (one for each isomorphism type). Simplifying, we may describe it as an iteration of the following: spinning up a non-zero vector $v \in V$ (see below) under the action of $\mathcal{A}_V$ yields

[1] For example, in the C–MeatAxe the size of an $r \times s$-matrix over a field of $q \leqslant 256$ elements is approximately $rs/(e10^9)$ gigabyte, where e is the largest integer such that $q^e \leqslant 256$, see [38].

a non-zero submodule $W = vA$ of V. If $W < V$, then δ is split into the representations of A on W and V/W. Otherwise, V might be irreducible. This can be proved (or disproved) by Norton's irreducibility criterion, see [42, (1.3.3)], that we state below. Recall that V^* denotes the dual space of V, whereas $V^{\circledast}$ denotes the dual (right) A-module of V. We may turn V^* into a left A-module by setting $v^{af} := (va)^f$ for all $v \in V$, $f \in V^*$, $a \in A$.

(3.1) Theorem (Norton). Let $a \in A$ such that $\ker a_V \neq 0$. Then V is irreducible if and only if $V = vA$ for all non-zero $v \in \ker a_V$ and $V^* = Av^*$ for some $v^* \in \ker a_{V^*}$. ∎

The program zsp can be used for individual spin-up and split procedures. Suppose that V is irreducible. By spinning up a non-zero vector $v \in V$ in a canonical order under $\mathcal{A}$ we obtain a *standard basis* $\mathcal{S}_v := (v_1, \ldots, v_d)$ of V such that $v_1 = v$ and $v_i = vb_i$ with a suitable product b_i over $\mathcal{A}$ for each $i \in \{2, \ldots, d\}$. If τ is an A-isomorphism from V to an A-module W, then $\mathcal{S}_v$ induces a standard basis $\mathcal{S}_{v^\tau} = (v_1^\tau, \ldots, v_d^\tau) = \mathcal{S}_v^\tau$ of W with $v^\tau b_i = (vb_i)^\tau = v_i^\tau$ for all $i \in \{2, \ldots, d\}$. As an important feature, we have

$$v_i a = \sum_{j=1}^{d} x_{ij} v_j \quad \text{if and only if} \quad v_i^\tau a = \sum_{j=1}^{d} x_{ij} v_j^\tau \quad \text{for all} \quad a \in A.$$

Thus ${}_{\mathcal{S}_v}[a]_{\mathcal{S}_v} = {}_{\mathcal{S}_{v^\tau}}[a]_{\mathcal{S}_{v^\tau}}$ for all $a \in A$. In particular, if we are given $\mathcal{A}_V$ and $\mathcal{A}_W$ subject to bases $\mathcal{B}$ of V and $\mathcal{C}$ of W, respectively, then

$${}_{\mathcal{S}_v}[a]_{\mathcal{S}_v} = B \cdot {}_{\mathcal{B}}[a]_{\mathcal{B}} \cdot B^{-1} = C \cdot {}_{\mathcal{C}}[a]_{\mathcal{C}} \cdot C^{-1} = {}_{\mathcal{S}_{v^\tau}}[a]_{\mathcal{S}_{v^\tau}} \quad \text{for all} \quad a \in \mathcal{A}$$

where $B = {}_{\mathcal{S}_v}[\mathrm{id}_V]_{\mathcal{B}}$ and $C = {}_{\mathcal{S}_{v^\tau}}[\mathrm{id}_W]_{\mathcal{C}}$ are the appropriate change of basis matrices from $\mathcal{S}_v$ to $\mathcal{B}$ and from $\mathcal{S}_{v^\tau}$ to $\mathcal{C}$, respectively. This provides a necessary condition for equivalence of representations. In the above form, however, it depends on the knowledge of τ, thus not being of much practical sense. To ensure the latter, let us consider $E := \mathrm{End}_A(V)$ which is a division algebra over F embedded naturally as $F \cdot \mathrm{id}_V \leqslant E$, thus an F-vector space of finite dimension $[E : F] := \dim_F E$. In particular, as F is finite, $E \geqslant F$ is a field extension of finite degree $[E : F]$. Moreover, as E is a field, its regular module is the only irreducible E-module (up to isomorphism). Therefore, if U is an E-module, then $U \cong m \cdot E$ with $m = \dim_E U$ and $\dim_F U = [E : F] \cdot m$. If U is irreducible, then $U \cong E$ and $E^\times$ acts transitively on the non-zero elements of U.

(3.2) Lemma. Let V be an irreducible A-module and $E := \mathrm{End}_A(V)$. If U is an E-module with $\dim_F U = [E : F]$, then $E^\times$ acts transitively on $U \setminus \{0\}$. In particular, if $U \subseteq V$, then the matrix representation of A on V with respect to a standard basis $\mathcal{S}_u$ for $u \in U \setminus \{0\}$ is independent of u. ∎

Let c be a *word* over $\mathcal{A}$, that is, an element of A written as an F-linear combination over $\mathcal{A}$. Then $\ker c_V$ is naturally an E-module with F-dimension $[E : F] \cdot \dim_E \ker c_V$. Assume that c is an *idword*, that is, $\dim_F \ker c_V = [E : F]$.#2 Then $\ker c_V$ is an irreducible E-module and $E^\times$ acts transitively on $\ker c_V \setminus \{0\}$. If V and W are isomorphic A-modules, then the same holds for $\ker c_W \cong_F \ker c_V$. In particular, choosing arbitrary non-zero elements $v \in \ker c_V$ and $w \in \ker c_W$, we get ${}_{\mathcal{S}_v}[a]_{\mathcal{S}_v} = {}_{\mathcal{S}_w}[a]_{\mathcal{S}_w}$ for all $a \in A$. Note that this explicitly constructs an A-isomorphism τ from V to W such that ${}_{\mathcal{B}}[\tau]_{\mathcal{C}} = B^{-1}C$ with change of basis matrices B and C as above. Conversely, the non-existence of such τ can be concluded from ${}_{\mathcal{S}_v}[a]_{\mathcal{S}_v} \neq {}_{\mathcal{S}_w}[a]_{\mathcal{S}_w}$ for some $a \in \mathcal{A}$. For information on how to find $[E : F]$ and idwords c, see [22, 38].
Similarly, we can make use of a standard basis if we are interested in realizing δ over some smaller field. This is particularly useful if F is a splitting field for δ, that is, $[E : F] = 1$.

(3.3) Lemma. Let $\delta : A \to F^{d\times d}$ be an irreducible matrix representation of A over F and suppose that c is an idword over $\mathcal{A}$ with coefficients solely from the prime field F_p of F. Let $\mathcal{S}_v = (v_1, \dots, v_d)$ be the standard basis of F^d with respect to $\mathcal{A}^\delta$ and some $v \in \ker c^\delta \setminus \{0\}$, and denote by $[\mathcal{S}_v]$ the $d \times d$-matrix whose ith row is v_i for $i \in \{1, \dots, d\}$. Then $[\mathcal{S}_v]\mathcal{A}^\delta[\mathcal{S}_v]^{-1}$ realizes δ over the smallest possible field $K \leqslant F$.

Proof. Let $K \leqslant F$ be minimal such that δ is realizable as δ' over K. Then δ and δ' may be considered as equivalent F-representations, both written with respect to the canonical basis of F^d. Thus $[\mathcal{S}_v]a^\delta[\mathcal{S}_v]^{-1} = [\mathcal{S}_w]a^{\delta'}[\mathcal{S}_w]^{-1} \in K^{d\times d}$ for all $a \in \mathcal{A}$ and arbitrary non-zero elements $v \in \ker c^\delta$, $w \in \ker c^{\delta'}$. ∎

Note that irreducible representations output by chop are always written with respect to a standard basis as in (3.3) and therefore over their field of definition.

Condensation

We follow [64] in the description of the basic facts on what is called *condensation*. Let V be an A-module. With respect to an idempotent $e \in A$, the eAe-module Ve is said to be the *condensed module* of V. Note that $1_{eAe} = e$, thus e acts as the identity on every eAe-module.

2The existence of such c follows from the double centralizer property $A^\delta \cong \mathrm{End}_E(V)$ since we can construct an element $\sigma \in \mathrm{End}_E(V)$ with $\dim_E \ker \sigma = 1$.

(3.4) Proposition. Let e be an idempotent of the F-algebra A and let V be an A-module.

(1) If W is an A-submodule of V, then We is an eAe-submodule of Ve.

(2) If W is an A-submodule of V, then $(V/W)e \cong Ve/We$ as eAe-modules.

(3) If U is an eAe-submodule of Ve, then $U = We$ for an A-submodule W of V.

(4) If V is irreducible, then either $Ve = 0$ or Ve is an irreducible eAe-module.

Proof. (1) Clearly, We is a subspace of Ve with $We(eAe) = (WeA)e \subseteq We$.
(2) Define an eAe-homomorphism $\varphi : Ve \to (V/W)e, v \mapsto (v + W)e = ve + W$. Each $ve+W \in (V/W)e$ has a preimage $ve \in Ve$ under φ, and it is easy to see that $\ker \varphi = We$.
(3) As U is a subspace of V, clearly $W := UA \subseteq VA = V$ is an A-submodule of V such that $U = Ue \subseteq UAe = We = UeAe = U$, thus $We = U$.
(4) If $U \neq 0$ is an eAe-submodule of Ve, then by (3) there is some A-submodule W with $U = We$. Hence $0 \neq W = V$ and $U = We = Ve$. ∎

(3.5) Theorem. Let e be an idempotent of the F-algebra A and let V be an A-module. If $0 = V_0 < V_1 < \cdots < V_n = V$ is a composition series of V, then there is a sequence $0 \leqslant i_0 < \cdots < i_m \leqslant n$ such that $0 = V_{i_0}e < V_{i_1}e < \cdots < V_{i_m}e = Ve$ is a composition series of Ve. In particular, if $\{X_1, \ldots, X_n\}$ are the A-composition factors of V, then $\{X_i e : X_i e \neq 0, i \in \{1, \ldots, n\}\}$ are the eAe-composition factors of Ve. Conversely, if $0 = W_0 < W_1 < \cdots < W_m = Ve$ is a composition series of Ve, then V has a composition series $0 = V_{01} < \cdots < V_{0r_0} < V_{11} < \cdots < V_{1r_1} < \ldots < V_{m1} < \cdots < V_{mr_m} = V$ such that $V_{ij}e \cong W_i$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, r_i\}$.

Proof. Let $0 = V_0 < V_1 < \cdots < V_n = V$ be a composition series of V. By (3.4), we have $0 = V_0e \leqslant V_1e \leqslant \cdots \leqslant V_ne = Ve$. For each $i \in \{1, \ldots, n\}$, we have $V_{i-1}e = V_ie$ if $(V_i/V_{i-1})e = V_ie/V_{i-1}e = 0$; otherwise, $V_ie/V_{i-1}e$ is irreducible.
Now suppose that $0 = W_0 < W_1 < \cdots < W_m = Ve$ is a composition series of Ve. We have $0 = W_0A \leqslant W_1A \leqslant \cdots \leqslant W_mA = VeA \leqslant VA = V$ with $(W_iA)e = W_i$ for all $i \in \{1, \ldots, m\}$. Refine this sequence to a composition series of V and let $i > 1$ be such that $W_{i-1}A \leqslant W \leqslant W_iA$ for an A-module W. It follows that $W_{i-1} \leqslant We \leqslant W_i$. As W_{i-1} is maximal in W_i, either $W_{i-1} = We$ or $We = W_i$, whence the claim follows. ∎

(3.6) Definition. An A-module V is called *local* if its head $\mathrm{hd}(V)$ is irreducible. In this case, V is S-*local* if $\mathrm{hd}(V) \cong S$. An idempotent $e \in A$ is S-*primitive* if eA is S-local.

Let $e \in A$ be an idempotent such that $\mathrm{hd}(eA) \cong m \cdot S$ is isomorphic to a direct sum of m copies of an irreducible A-module S. Decomposing $e = \sum_{i=1}^{m} e_i$ as a sum of pairwise orthogonal S-primitive idempotents e_i, it follows that

$$Se \cong \mathrm{Hom}_A(eA, S) \cong \bigoplus_{i=1}^{m} \mathrm{Hom}_A(e_i A, S) \cong \bigoplus_{i=1}^{m} \mathrm{End}_A(S)$$

are isomorphic F-vector spaces. Thus $\dim_F Se = m \cdot \dim_F \mathrm{End}_A(S)$. If e is S-primitive, that is $m = 1$, then $\mathrm{End}_A(S)^\times$ acts transitively on the non-zero elements of Se, see (3.2).

(3.7) Theorem ([39, (2.3)]). Let $e \in A$ be an S-primitive idempotent and $\mathcal{V} := Ve \setminus \{0\}$ for an A-module V.

(1) The S-local A-submodules of V are exactly the A-modules vA for $v \in \mathcal{V}$.

(2) The local eAe-submodules of Ve are exactly the eAe-modules $v \cdot eAe$ for $v \in \mathcal{V}$. ∎

Recall that a_V denotes the endomorphism of V induced by $a \in A$. The *Fitting projection* $\mathrm{fit}_a(V)$ from V onto $\ker a_V^n$ with respect to the complement $\mathrm{im}\, a_V^n$ is given by the Fitting decomposition $V = \ker a_V^n \oplus \mathrm{im}\, a_V^n$ as vector spaces.

(3.8) Definition. Let V be an A-module with irreducible constituent S. An S-*peakword* for V is an element $a \in A$ such that $\dim_F \ker a_S^2 = \dim_F \mathrm{End}_A(S)$ and $\ker a_T = 0$ for all irreducible constituents $T \not\cong S$ of V.

(3.9) Theorem. Let V be a faithful A-module. For each $a \in A$ there exists an uniquely determined idempotent $e \in A$ that induces the Fitting projection on V with respect to a, that is, $Ve = \mathrm{fit}_a(V)$. If S is an irreducible constituent of V, then $Se = \mathrm{fit}_a(S)$. Furthermore, if a is an S-peakword for V, then e is S-primitive.

Proof. See Theorems 3.1, 3.2, and 3.4 in [39]. ∎

Recall from above that $\dim_F \mathrm{End}_A(S) \mid \dim_F \ker a_S$ for an irreducible A-module S and all $a \in A$. In the situation of the preceding theorem with an S-peakword a for V, we deduce that $\dim_F \ker a_S^2 = \dim_F \mathrm{End}_A(S) = \dim_F \ker a_S$. Thus the Fitting decomposition of S

with respect to a is $\ker a_S \oplus \operatorname{im} a_S$. We note that the idempotent e that induces $\mathrm{fit}_a(V)$ for $a \in A$ is constructed by evaluating a certain polynomial at a_V, see [39]. This allows "direct condensation" of V by determination of the stable nullspace $Ve = \ker a_V^n$, followed by determining the action of eb_Ve for $b \in A$ on Ve. Perhaps the main importance of peakwords is that they provide primitive idempotents which give rise to local submodules of Ve (see (3.7)). These in turn can be used to determine the whole submodule lattice of V, see [39].

Fixed point condensation of tensor products

For applications in the context of this work, consider the group algebra FG again. Let K be a subgroup of G whose order is prime to $p = \operatorname{char} F$. We refer to K as a *condensation subgroup* of G that induces the *trivial* idempotent

$$e_K := |K|^{-1} \sum_{k \in K} k \in FG,$$

belonging to the trivial character 1_K of K. Set $e := e_K$ for short. If V is an FG-module with Brauer character ϕ, then $Ve = \mathrm{Fix}_V(K)$ is the F-subspace of all K-fixed points of V. Its dimension is just the multiplicity of the trivial FK-module in $V{\downarrow_K}$, that is, $\dim_F Ve = \langle \phi{\downarrow_K}, 1_K \rangle$ as FK is semisimple.

We make use of condensation for tensor products $V \otimes_F W$ of FG-modules V and W with respect to e, see [43]. This method determines the action of given elements $ege \in eFGe$ on a suitable basis of $(V \otimes W)e$ *without* determining the action of g on $V \otimes W$ before.

(3.10) Theorem. Let V and W be FG-modules. Assume that I is an index set of tuples (i, j) such that for each $(i, j) \in I$ there are irreducible constituents S_i of $V{\downarrow_K}$ and T_j of $W{\downarrow_K}$ with $S_i^{\circledast} \cong T_j$, their multiplicities being m_i and n_j, respectively. Then

$$(V \otimes W)e \cong \bigoplus_{(i,j) \in I} m_i n_j \cdot (S_i \otimes T_j)e.$$

Proof. See [43, (1.2)]. The claim follows from the semisimplicity of FK, the distributivity of the tensor product, and the fact that for irreducible FK-modules S and T the FK-isomorphism $S \otimes T \cong \mathrm{Hom}_F(S^{\circledast}, T)$ implies $(S \otimes T)e \cong \mathrm{Hom}_{FK}(S^{\circledast}, T)$, the latter being non-zero if and only if $T \cong S^{\circledast}$. ∎

In order to utilize the reduction to blocks of the form $(S \otimes S^{\circledast})e$ for an irreducible FK-

module S imposed by (3.10), suitable bases for V and W reflecting their reducibility as FK-modules are needed.

(3.11) Definition. Let A be an F-algebra and let U be an A-module that is isomorphic to a direct sum of pairwise non-isomorphic irreducible A-modules $U_1, \dots, U_n$ where U_i occurs with multiplicity m_i for $i \in \{1, \dots, n\}$. Denote by $\rho_i : A \to F^{d_i \times d_i}$ a given matrix representation of A on U_i. A basis $\mathcal{B}$ of U such that for every $a \in A$ the representation matrix with respect to $\mathcal{B}$ on U is a block diagonal matrix of the form

$$_{\mathcal{B}}[a]_{\mathcal{B}} = \mathrm{diag}(\underbrace{a^{\rho_1}, \dots, a^{\rho_1}}_{m_1\text{-times}}, \dots, \underbrace{a^{\rho_n}, \dots, a^{\rho_n}}_{m_n\text{-times}})$$

is called a *symmetry basis* of U.

There are different approaches to compute symmetry bases, see [43, 57]. One of them that is currently used in the C–MeatAxe is based on peakwords, and can be used more generally to compute the socle series of a given module, see [44]. In (3.14) below, we describe a method for computing a symmetry basis in case of an abelian condensation subgroup ([53]). Here, let us assume that

$$V{\downarrow_K} = \bigoplus_{i=1}^{s} \bigoplus_{\alpha=1}^{m_i} S_{i\alpha} \quad \text{and} \quad W{\downarrow_K} = \bigoplus_{j=1}^{t} \bigoplus_{\beta=1}^{n_j} T_{j\beta}$$

with respect to symmetry bases

$$\mathcal{B} = \bigsqcup_{i=1}^{s} \bigsqcup_{\alpha=1}^{m_i} \mathcal{B}_{i\alpha} \quad \text{and} \quad \mathcal{C} = \bigsqcup_{j=1}^{t} \bigsqcup_{\beta=1}^{n_j} \mathcal{C}_{j\beta}$$

of $V{\downarrow_K}$ and $W{\downarrow_K}$, respectively. Here it is to be understood that $\mathcal{B}_{i\alpha}$ and $\mathcal{C}_{j\beta}$ are bases of the irreducible FK-modules $S_{i\alpha}$ respectively $T_{j\beta}$, and $S_{i1} \cong \cdots \cong S_{im_i}$ as well as $T_{j1} \cong \cdots \cong T_{jn_j}$ for all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$. Then

$$(V \otimes W){\downarrow_K} = \bigoplus_{i=1}^{s} \bigoplus_{\alpha=1}^{m_i} \bigoplus_{j=1}^{t} \bigoplus_{\beta=1}^{n_j} S_{i\alpha} \otimes T_{j\beta}$$

with respect to

$$\mathcal{D} := \mathcal{B} \otimes \mathcal{C} = \bigsqcup_{i=1}^{s} \bigsqcup_{\alpha=1}^{m_i} \bigsqcup_{j=1}^{t} \bigsqcup_{\beta=1}^{n_j} \mathcal{B}_{i\alpha} \otimes \mathcal{C}_{j\beta}$$

where $\mathcal{B}_{i\alpha} \otimes \mathcal{C}_{j\beta}$ denotes the lexicographical tensor product basis of $S_{i\alpha} \otimes T_{j\beta}$. Using the equivalence test based on standard bases that we described above, we can find the set

$$I := \{ (i,j) : i \in \{1,\dots,s\}, j \in \{1,\dots,t\}, S_{i1}^{\circledast} \cong T_{j1} \}.$$

By (3.10), we have

$$(V \otimes W)e = \bigoplus_{(i,j)\in I} \bigoplus_{\alpha=1}^{m_i} \bigoplus_{\beta=1}^{n_j} (S_{i\alpha} \otimes T_{j\beta})e.$$

For irreducible FK-modules S and T with $S^{\circledast} \cong T$ and bases B of S and C of T, consider the matrix $E := {}_D[e]_D$ of e on $S \otimes T$ with respect to $D := B \otimes C$. Then E is given by evaluating e on $S \otimes T$ in terms of Kronecker products of the matrices induced by the action of K on S and T, that is,

$$E = |K|^{-1} \sum_{k \in K} {}_B[k]_B \otimes {}_C[k]_C.$$

Now a basis $\mathfrak{q}$ of the row space of E may be interpreted as the coefficient matrix that expresses a basis Q of $(S \otimes T)e$ in terms of D. In symbols, $\mathfrak{q} = {}_Q[e]_D$ is the matrix of the embedding $(S \otimes T)e \to S \otimes T$ with respect to the bases Q and D. Vice versa, let $\mathfrak{p} = {}_D[e]_Q$ be the matrix of the e-projection $S \otimes T \to (S \otimes T)e$ with respect to D and Q. Then $\mathfrak{p}\mathfrak{q} = {}_D[e]_Q\,{}_Q[e]_D = {}_D[e^2]_D = E$ and $\mathfrak{q}\mathfrak{p} = {}_Q[e]_D\,{}_D[e]_Q = {}_Q[e^2]_Q$ is the identity matrix of size $|Q|$ since e acts as the identity on $(S \otimes T)e$. Note that $\mathfrak{p}$ is uniquely determined by $\mathfrak{q}$ and can be obtained straightforwardly while $\mathfrak{q}$ is computed from E. If $v \in S \otimes T$, then the coefficient vector of ve with respect to Q is $ve_Q = v_D \cdot {}_D[e]_Q = v_D \cdot \mathfrak{p}$. Moreover, the action of $ege \in eFGe$ on $(S \otimes T)e$ with respect to Q is given by

$${}_Q[ege]_Q = {}_Q[e]_D\,{}_D[g]_D\,{}_D[e]_Q = \mathfrak{q}\,({}_B[g]_B \otimes {}_C[g]_C)\,\mathfrak{p}.$$

Set $D_{(i,j)\alpha\beta} := \mathcal{B}_{i\alpha} \otimes \mathcal{C}_{j\beta}$ and let $\mathfrak{q}_{(i,j)}$ be a basis of the row space of

$$E_{(i,j)} := |K|^{-1} \sum_{k \in K} {}_{\mathcal{B}_{i1}}[k]_{\mathcal{B}_{i1}} \otimes {}_{\mathcal{C}_{j1}}[k]_{\mathcal{C}_{j1}}.$$

As above, there is a basis $Q_{(i,j)\alpha\beta}$ of $(S_{i\alpha} \otimes T_{j\beta})e$ such that $\mathfrak{q}_{(i,j)}$ is the matrix of e with respect to $Q_{(i,j)\alpha\beta}$ and $D_{(i,j)\alpha\beta}$ for all $(i,j) \in I$, $\alpha \in \{1,\dots,m_i\}$, $\beta \in \{1,\dots,n_j\}$. Also let

$\mathfrak{p}_{(i,j)}$ denote the unique matrix such that $E_{(i,j)} = \mathfrak{p}_{(i,j)}\mathfrak{q}_{(i,j)}$ for $(i,j) \in I$. Then

$$\mathcal{Q} := \bigsqcup_{(i,j)\in I} \bigsqcup_{\alpha=1}^{m_i} \bigsqcup_{\beta=1}^{n_j} Q_{(i,j)\alpha\beta}$$

is a basis of $(V \otimes W)e$ and the matrix of $ege \in eFGe$ on $(V \otimes W)e$ with respect to $\mathcal{Q}$ can be determined blockwise: the block $\square$ at row coordinate $((i,j),\alpha,\beta)$ and column coordinate $((k,l),\gamma,\delta)$ is

$$_{Q_{(i,j)\alpha\beta}}[ege]_{Q_{(k,l)\gamma\delta}} = \mathfrak{q}_{(i,j)} \left(_{\mathcal{B}_{i\alpha}}[g]_{\mathcal{B}_{k\gamma}} \otimes {}_{\mathcal{C}_{j\beta}}[g]_{\mathcal{C}_{l\delta}} \right) \mathfrak{p}_{(k,l)}.$$

In this way, the complete matrix of ege is

$$\boxplus_{(i,j)\in I} \boxplus_{\alpha=1}^{m_i} \boxplus_{\beta=1}^{n_j} \boxplus_{(k,l)\in I} \boxplus_{\gamma=1}^{m_k} \boxplus_{\delta=1}^{n_l} {}_{Q_{(i,j)\alpha\beta}}[ege]_{Q_{(k,l)\gamma\delta}}.$$

Furthermore, for $v \in V \otimes W$ the condensed vector ve with respect to $\mathcal{Q}$ is

$$ve_{\mathcal{Q}} = \bigsqcup_{(i,j)\in I} \bigsqcup_{\alpha=1}^{m_i} \bigsqcup_{\beta=1}^{n_j} ve_{Q_{(i,j)\alpha\beta}} = \bigsqcup_{(i,j)\in I} \bigsqcup_{\alpha=1}^{m_i} \bigsqcup_{\beta=1}^{n_j} ve_{D_{(i,j)\alpha\beta}} \cdot \mathfrak{p}_{(i,j)},$$

and $ve_{\mathcal{Q}}$ is embedded into $V \otimes W$ again as

$$ve_{\mathcal{D}} = \bigsqcup_{i=1}^{s} \bigsqcup_{\alpha=1}^{m_i} \bigsqcup_{j=1}^{t} \bigsqcup_{\beta=1}^{n_j} ve_{D_{(i,j)\alpha\beta}}$$

where $ve_{D_{(i,j)\alpha\beta}}$ is zero if $(i,j) \notin I$ and $ve_{D_{(i,j)\alpha\beta}} = ve_{Q_{(i,j)\alpha\beta}} \cdot \mathfrak{q}_{(i,j)}$ otherwise.
For further details, see [43]. The method of condensing a tensor product with respect to the trivial idempotent of K was generalized in [57] to arbitrary linear idempotents induced by K.

The generation problem

Drawing conclusions from (3.4) and (3.5) in the practical application of condensation is often complicated by the *generation problem* of proving that a given subset $X \subset eAe$ is a set of generators for eAe. Note that in general $e\mathcal{A}e$ does not generate eAe even if $\mathcal{A}$ generates A. Criteria that ensure the generation of eAe were studied in [57], see also [38, 58]. In our applications, however, we do not strive for proving generation of the whole *Hecke algebra* eAe but work with a subalgebra $C = \langle X \rangle$ of eAe, the *condensation algebra*, instead. That is, we consider the restriction of a condensed module $Ve\downarrow_C$ rather than the eAe-module Ve itself. We remark that, in turn, this sometimes requires a delicate analysis of $Ve\downarrow_C$.

Further methods

Let $\mathcal{G}$ be a list of generators of G and $\delta : G \to GL_d(F)$ be a matrix representation of G where $F \leqslant f$ with f as in Section 1. Under the assumption of computational feasibility, if representatives $g_1, \ldots, g_l$ of the p-regular classes of G (or at least those in question) are known as words over $\mathcal{G}$, then the Brauer character of δ may be computed as described in Section 1. Sometimes this is also useful in the following situation.

(3.12) Assume that we are in the situation (b) of (1.6) such that δ is irreducible and we know its Brauer character φ. The irreducible constituents ν, ν^c of $\varphi\downarrow_N = \nu + \nu^c$ may be obtained as follows, provided we know the class fusion of $N_{p'}$ in $G_{p'}$. Denote conjugacy in N by $\sim_N$. For some $g \in G \setminus N$, build up the list $\mathcal{N} := (g_i^\delta : i \in \{1, \ldots, l\}, g_i \in N, g_i \nsim_N g_i^g)$ and extend $\mathcal{N}$ such that $\langle \mathcal{N} \rangle = N^\delta$ if need be. By definition, for $g_i \in N$ with $g_i \nsim_N g_i^g$, we have $\nu^c(g_i) = \nu(g_i^g)$ and $\nu(g_i) = \nu^c(g_i^g)$. Suppose that the irreducible constituents ε^+ and ε^- of $\delta\downarrow_N$ with respect to $\mathcal{N}$ are known. Then, without loss, for all $g_i \in N$ with $g_i \nsim_N g_i^g$ we may assign the Brauer character value of $g_i^{\varepsilon^+}$ to $\nu(g_i)$, and the Brauer character value of $g_i^{\varepsilon^-}$ to $\nu^c(g_i)$. Moreover, for all $g_i \in N$ with $g_i \sim_N g_i^g$, clearly $\nu(g_i) = \nu^c(g_i) = \varphi(g_i)/2$. ∎

Let K be a p'-subgroup of G with trivial idempotent e. In order to employ the following *trace formula*, on one hand we need to evaluate the trace of a matrix of an element exe for $x \in G$ on a condensed module Ve, on the other hand we need to know the distribution of the elements from the coset xK into the conjugacy classes of G. We have seen above how the former task can be handled, for example, if V is contained in a suitable tensor product of FG-modules. In Section 5, we present a way to solve the latter task conveniently in case that $G \in \{\tilde{S}_n, \tilde{A}_n\}$.

(3.13) Let V be an FG-module that affords the Brauer character ϕ, and denote by tr_V and tr_{Ve} the F-valued trace functions corresponding to V and Ve, respectively. Since $V = Ve \oplus V(1-e)$ as vector spaces, we have $\mathrm{tr}_V(exe) = \mathrm{tr}_{Ve}(exe)$ for $x \in G$, which implies

$$\mathrm{tr}_{Ve}(exe) = \mathrm{tr}_V(exe) = \mathrm{tr}_V(xe^2) = \mathrm{tr}_V(xe) = |K|^{-1} \sum_{k \in K} \mathrm{tr}_V(xk).$$

Setting $\check{\phi}(g) := \phi(g_{p'})$, we have $\overline{\check{\phi}(g)} = \mathrm{tr}_V(g)$ for all $g \in G$. Therefore, for all $x \in G$,

$$\mathrm{tr}_{Ve}(exe) = |K|^{-1} \sum_{k \in K} \overline{\check{\phi}(xk)}. \qquad \blacksquare$$

Returning to the discussion of condensing tensor product modules $V \otimes W$, the following is sometimes useful for computing symmetry bases of $V{\downarrow_K}$ and $W{\downarrow_K}$.

(3.14) Assume that K is abelian and let $\mathcal{I}(K)$ be the set of irreducible characters of FK. (Under the assumption that $\mathrm{Irr}(K)$ is known, the elements of $\mathcal{I}(K)$ are the sums over the orbits of algebraically conjugate elements of $\mathrm{IBr}(K) = \{\overline{\chi} : \chi \in \mathrm{Irr}(K)\}$ that fuse over F.[#3]) Since FK is semisimple and commutative, we have $FK \cong \bigoplus_{\chi \in \mathcal{I}(K)} e_\chi FK$ with centrally primitive orthogonal idempotents

$$e_\chi = \sum_{k \in K} \chi(k^{-1})k$$

such that $e_\chi FK \cong \mathrm{End}_{FK}(U_\chi) =: E_\chi$ where U_χ denotes an irreducible FK-module that affords $\chi \in \mathcal{I}(K)$. Moreover, $\dim_F U_\chi = [E_\chi : F]$ and $(E_\chi)^\times$ acts transitively on $U_\chi \setminus \{0\}$ for all $\chi \in \mathcal{I}(K)$. Let U be an FK-module with matrix representation δ. Then $U \cong \bigoplus_{\chi \in \mathcal{I}(K)} m_\chi \cdot U_\chi$ with $m_\chi \in \mathbb{Z}^+$ and for $\chi \in \mathcal{I}(K)$ the row vectors of $(e_\chi)^\delta$ are generators for $Ue_\chi \cong m_\chi \cdot U_\chi$. Note that we have to sum over all elements of K^δ to build up $(e_\chi)^\delta$. Now the transitive action of $(E_\chi)^\times$ on the non-zero elements of U_χ (see (3.2)) induces a transitive action of $\mathrm{End}_{FK}(Ue_\chi)^\times \cong (E_\chi^{m_\chi \times m_\chi})^\times$ on $Ue_\chi \setminus \{0\}$. Begin with the empty sequence $\mathcal{B}_\chi$. For each row vector u of $(e_\chi)^\delta$, if $u \notin \langle \mathcal{B}_\chi \rangle$, append the standard basis $\mathcal{S}_u$ with respect to u and $(FK)^\delta$ to $\mathcal{B}_\chi$. We need to show that the matrix representation of FK with respect to $\mathcal{S}_u$ is independent of u: provided $Ue_\chi \neq 0$, there is a submodule $W \leqslant Ue_\chi$ with $W \cong U_\chi$. Since $\mathrm{End}_{FK}(W)^\times$ acts transitively on $W \setminus \{0\}$, the matrix representation of FK with respect to a standard basis $\mathcal{S}_w$ for any non-zero element $w \in W$ is independent of w. Thus, the matrix representation of FK with respect to $\mathcal{S}_u$ is the same as the one with respect to $\mathcal{S}_w$ for any non-zero $w \in W$ by the transitivity of $\mathrm{End}_{FK}(Ue_\chi)^\times$ on $Ue_\chi \setminus \{0\}$. Consequently, we obtain a symmetry basis of Ue_χ, and by iteration over the non-zero elements of $\{(e_\chi)^\delta : \chi \in \mathcal{I}(K)\}$ we obtain a symmetry basis of U. $\blacksquare$

Finally, we note that often there is no standardized and at the same time feasible way of constructing a particular representation for G. However, having at least one faithful representation at hand, "larger representations are made by tensoring smaller ones, smaller

[3]See `RealizableBrauerCharacters()` in GAP.

representations are obtained as constituents of larger ones" ([61, p. 267]). In order to ease the reproduction of our results, we use the following terminology for documenting the way a particular representation was obtained. Given representations D_0, E_0 of G, to *chop the sequence*

$$(D_0 \otimes E_0, D_1 \otimes E_1, \ldots, D_r \otimes E_r)$$

means that D_i, E_i are among the irreducible constituents of $D_{i-1} \otimes E_{i-1}$ or elements of $\{D_0, E_0\}$ for $i \in \{1, \ldots, r\}$, the representation of interest being a constituent of $D_r \otimes E_r$. Similarly, if $G_0 \leqslant G_1 \leqslant \ldots \leqslant G_s = G$ and D_0 is some representation of G_0, to chop the sequence

$$(D_0 {\uparrow^{G_1}}, D_1 {\uparrow^{G_2}}, \ldots, D_{s-1} {\uparrow^{G_s}})$$

means that D_i is an irreducible constituent of $D_{i-1} {\uparrow^{G_i}}$ for $i \in \{1, \ldots, s-1\}$ and the representation D_s that we are interested in occurs as a constituent of $D_{s-1} {\uparrow^{G}}$.

4 Graded representation theory of $\tilde{S}_n$

We introduce some terminology from the superalgebra framework that seems to be the right setting for the representation theory of $\tilde{S}_n$. Essentially it is about $\mathbb{Z}_2$-graded objects. Our exposition follows [29, 32] but is restricted to the notions that will be needed subsequently.

4.1 Notions of superalgebra

Throughout this section, we shall sensibly assume that F is a field of characteristic different from 2.

Superspaces

A *superspace* over F is a $\mathbb{Z}_2$-graded finite dimensional F-vector space $V = V_0 \oplus V_1$, where $\mathbb{Z}_2 = \{0, 1\}$. An element $v \in V_i$ is called *homogeneous* of *degree* $d(v) = i \in \mathbb{Z}_2$, and we denote by $h(V) := V_0 \cup V_1$ the set of all homogeneous elements of V. We write $|V|$ if we "desuperize" V, that is, if we consider V without its graded structure.
A subspace U of $|V|$ is *homogeneous* if $U = (U \cap V_0) + (U \cap V_1)$, in which case by definition U is a *subsuperspace* of V and the sum is direct. We define an involutory linear map $\delta_V : V \to V$ by linear extension of $v \mapsto (-1)^{d(v)} v$ for $v \in h(V)$.

(4.1.1) Lemma. Let V be a superspace over F. A subspace U of $|V|$ is a subsuperspace of V if and only if $U^{\delta_V} \subseteq U$.

Proof. Note that $\delta_V|_{U \cap V_i} = (-1)^i \mathrm{id}_{U \cap V_i}$ for $i \in \mathbb{Z}_2$. Clearly, if U is homogeneous, then $U^{\delta_V} = U$. For $u \in U$, let $v_0 \in V_0$ and $v_1 \in V_1$ be suitable elements such that $u = v_0 + v_1$. If we assume that $u^{\delta_V} = v_0 - v_1 \in U$, then we have $v_0 = 2^{-1}(u + u^{\delta_V}) \in U$ and $v_1 = 2^{-1}(u - u^{\delta_V}) \in U$, too. ∎

Let V and W be superspaces over F. The direct sum $V \oplus W$ of vector spaces becomes a superspace by grading $(V \oplus W)_i := V_i \oplus W_i$ for $i \in \mathbb{Z}_2$. Moreover, we call a linear map $\theta : V \to W$ *homogeneous* of *degree* $d(\theta) = i \in \mathbb{Z}_2$ if $V_j^\theta \subseteq W_{i+j}$ for all $j \in \mathbb{Z}_2$. Then we view $\mathrm{Hom}_F(V, W)$ as a superspace whose graded components are

$$\mathrm{Hom}_F(V, W)_i := \{\theta \in \mathrm{Hom}_F(V, W) : d(\theta) = i\} \quad (i \in \mathbb{Z}_2).$$

By setting $F_0 := F$ and $F_1 := 0$, F becomes a superspace. Hence we also may consider the *dual superspace* $V^* := \mathrm{Hom}_F(V, F) = \mathrm{Hom}_F(V_0, F) \oplus \mathrm{Hom}_F(V_1, F)$ of V.

Superalgebras and supermodules

A $\mathbb{Z}_2$-graded F-algebra $A = A_0 \oplus A_1$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$ is called a *superalgebra* over F. Sometimes we refer to the usual (ungraded) algebra by $|A|$. Let A and B be superalgebras over F. A *homomorphism* between A and B is defined as an algebra homomorphism $\sigma : |A| \to |B|$ subject to $d(\sigma) = 0$. Also an antiautomorphism τ of $|A|$ is an *antiautomorphism* of A if $d(\tau) = 0$.

A *right* A-*supermodule* is an F-superspace V together with a right action of A on V such that $V_i A_j \subseteq V_{i+j}$ for all $i, j \in \mathbb{Z}_2$. *Left* A-supermodules are defined analogously. A homogeneous linear map $\phi : V \to W$ of A-supermodules V and W is said to be an A-*homomorphism* of degree $d(\phi)$ if

$$(va)^{\phi} = (-1)^{d(\phi)d(a)} v^{\phi} a$$

for all $v \in V$ and $a \in h(A)$. By linear extension we obtain the superspace $\mathrm{Hom}_A(V, W)$ of all A-homomorphisms from V to W.

Let τ be an antiautomorphism of A, and let V be an A-supermodule. Then the dual superspace V^* of V can be given the structure of a (right) A-supermodule by setting $v^{fa} := (va^{\tau})^f$ for all $v \in V$, $f \in V^*$, $a \in A$. This A-supermodule is denoted by V^{τ}.

Finally, the notions of A-*subsupermodules*, *irreducibility*, and *complete reducibility* of A-supermodules are defined analogously as for A-modules. In particular, by (4.1.1), a subspace of an A-supermodule V is an A-subsupermodule if and only if it is invariant under δ_V and the action of A.

(4.1.2) Lemma. Let V and W be A-supermodules, where A is a superalgebra over F.

(1) $|\mathrm{Hom}_A(V, W)|$ and $\mathrm{Hom}_A(|V|, |W|)$ are isomorphic F-vector spaces.

(2) If $\phi \in \mathrm{Hom}_A(V, W)$ is homogeneous, then $\ker \phi$ is an A-subsupermodule of V and V^{ϕ} is an A-subsupermodule of W.

Proof. (1) For a homogeneous $\varphi \in \mathrm{Hom}_A(V,W)$ define $\varphi^- : V \to W$ by linear extension of $v \mapsto (-1)^{d(\varphi)d(v)} v^\varphi$ for $v \in h(V)$. Then an F-isomorphism is given by extending $\varphi \mapsto \varphi^-$ linearly to $\mathrm{Hom}_A(V,W)$.
(2) Let ϕ be homogeneous of degree $i \in \mathbb{Z}_2$. If $k \in K := \ker\phi$, then $k = v_0 + v_1$ for suitable $v_0 \in V_0$, $v_1 \in V_1$ such that $v_0^\phi = -v_1^\phi \in W_0 \cap W_1 = 0$. Thus $K^{\delta_V} \subseteq K$, so K is a subsuperspace of V by (4.1.1). Now K is A-invariant as $|K|$ is, hence K is an A-subsupermodule of V. Similarly, if $v^\phi = w_i + w_{i+1} \in V^\phi$ for suitable elements $w_i = v_0^\phi \in W_i$ and $w_{i+1} = v_1^\phi \in W_{i+1}$, then $(v^{\delta_V})^\phi = (-1)^i (v^\phi)^{\delta_W}$. Thus V^ϕ is invariant under δ_W, and the claim follows. ∎

(4.1.3) Definition. Let A be a superalgebra and let V be an irreducible A-supermodule. Then V is of *type* M if the A-module $|V|$ is irreducible. Otherwise, that is, if V has a proper non-homogeneous A-invariant subspace, V is of *type* Q.

Let V be an F-superspace with $\dim V_0 = m$, $\dim V_1 = n$ and consider its endomorphism superalgebra $\mathcal{M}(V) := \mathrm{End}_F(V)$. Pick a basis $\{v_{i,j}\}$ of V_i for $i \in \mathbb{Z}_2$, and let $M_{r,s}$ denote the set of $r \times s$-matrices over F ($r, s \in \mathbb{N}$). We may identify $\mathcal{M}(V)$ with the superalgebra

$$\mathcal{M}_{m,n} := \left\{ \left[\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right] : a \in M_{m,m}, d \in M_{n,n} \right\} \oplus \left\{ \left[\begin{smallmatrix} & b \\ c & \end{smallmatrix}\right] : b \in M_{m,n}, c \in M_{n,m} \right\}.$$

Then V is naturally an $\mathcal{M}_{m,n}$-supermodule of type M.
Suppose that $m = n$ and F contains a primitive fourth root of unity ω. There is another superalgebra $\mathcal{Q}_n \leqslant \mathcal{M}_{n,n}$ that acts on V: the homogeneous elements $X \in h(\mathcal{Q}_n)$ are characterized by $XJ = (-1)^{d(X)} JX$ for $J := \omega \left[\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right]$, that is, $(\mathcal{Q}_n)_0 := \left\{ \left[\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right] : a \in M_{n,n} \right\}$ and $(\mathcal{Q}_n)_1 := \left\{ \left[\begin{smallmatrix} & b \\ b & \end{smallmatrix}\right] : b \in M_{n,n} \right\}$. Although V is an irreducible $\mathcal{Q}_n$-supermodule, there are $|\mathcal{Q}_n|$-submodules $\langle v_{0,j} + v_{1,j} : j = 1, \ldots, n \rangle$ and $\langle v_{0,j} - v_{1,j} : j = 1, \ldots, n \rangle$ that are non-homogeneous subspaces of V. Thus V is a $\mathcal{Q}_n$-supermodule of type Q.

(4.1.4) Proposition. Assume that F is algebraically closed and let A be a superalgebra over F. If V is an A-supermodule of type Q, then there exist bases $\{v_{i,j}\}$ of V_i, $i \in \mathbb{Z}_2$, such that $|\{v_{0,j}\}| = |\{v_{1,j}\}| =: n \in \mathbb{N}$ and

$$|V| = \langle v_{0,j} + v_{1,j} : j = 1, \ldots, n \rangle \oplus \langle v_{0,j} - v_{1,j} : j = 1, \ldots, n \rangle$$

is a direct sum of two non-isomorphic irreducible A-modules. We denote these by V^+ and V^-. Moreover, the map $J_V : V \to V$ given by linear extension of $v_{0,j} \mapsto -v_{1,j}$ and $v_{1,j} \mapsto v_{0,j}$ for $j \in \{1, \ldots, n\}$ is an A-supermodule endomorphism of V with $d(J_V) = 1$.

Proof. In view of (4.1.1), there is an irreducible A-submodule U of $|V|$ such that $U^{\delta_V} \neq U$. As $u^{\delta_V} a = (u a^{\delta_A})^{\delta_V}$ for all $u \in U$ and $a \in A$, we see that U^{δ_V} is an irreducible A-submodule of $|V|$ too. Then $U \cap U^{\delta_V}$ and $U + U^{\delta_V}$ are invariant under δ_V and the action of A. Thus $U \cap U^{\delta_V} = 0$ and $V = U \oplus U^{\delta_V}$ since V is irreducible.
Let $\{u_1, \ldots, u_n\}$ be a basis of U, and set $v_{0,j} := u_j + u_j^{\delta_V}$, $v_{1,j} := u_j - u_j^{\delta_V}$ for $j \in \{1, \ldots, n\}$. Let $i \in \mathbb{Z}_2$. As $d(v_{i,j}) = i$, we have n linearly independent elements $v_{i,1}, \ldots, v_{i,n} \in V_i$. If $\dim_F V_i > n$, then $\dim_F V_{i+1} < n$, a contradiction. Thus $\{v_{i,1}, \ldots, v_{i,n}\}$ is a basis of V_i.
Let $\mathcal{E} := \mathrm{End}_A(V)$ and suppose that $U \cong U^{\delta_V}$ as A-modules. It follows from part (1) of (4.1.2) that $\dim_F \mathcal{E} = 4$, hence $\dim_F \mathcal{E}_i \geqslant 2$ for some $i \in \mathbb{Z}_2$. By part (2) of (4.1.2), we have $\ker \phi = 0$ for all $\phi \in \mathcal{E}_i \backslash \{0\}$, hence every such ϕ has an inverse ϕ^{-1}, and it is easy to see that ϕ^{-1} is contained in $\mathcal{E}_i$. In particular, $\mathcal{E}_i$ is a division algebra over F. For $e \in \mathcal{E}_i$, consider the F-linear map $\epsilon : \mathcal{E}_i \to \mathcal{E}_i$, $d \mapsto ed$. Since we assume F to be algebraically closed, ϵ has an eigenvalue $\lambda \in F$ such that $d^{\epsilon} = ed = \lambda d$ for some $d \in \mathcal{E}_i$. Thus $e = \lambda \, \mathrm{id}_V$ and $\mathcal{E}_i \cong F$, a contradiction. Finally, by (4.1.2).(1), J_V is an A-endomorphism of V since $J_V^- = -\mathrm{id}_U \oplus \mathrm{id}_{U^{\delta_V}} \in \mathrm{End}_A(|V|)$. We conclude that $\dim_F \mathcal{E} = 2$ with $J_V \in \mathcal{E}_1$. ∎

Now Schur's Lemma for A-modules and the proof of (4.1.4) show:

(4.1.5) Schur's Lemma. Assume that F is algebraically closed. Let A be a superalgebra over F and let V be an A-supermodule. If V is of type $\mathtt{M}$, then $\mathrm{End}_A(V) = \langle \mathrm{id}_V \rangle \cong F$. If V is of type $\mathtt{Q}$, then $\mathrm{End}_A(V)_0 = \langle \mathrm{id}_V \rangle \cong F$ and $\mathrm{End}_A(V)_1 = \langle J_V \rangle \cong F$. ∎

Let A be an algebra over F and let V be an A-module. Suppose that A has the structure of a superalgebra over F. Then $\delta := \delta_A$ is an algebra automorphism of A that gives rise to the twisted A-module V^{δ}: by definition, V^{δ} is the vector space V on which A acts by $v \cdot a := v a^{\delta}$ for $v \in V$, $a \in A$. Clearly, V^{δ} is irreducible if and only if V is.
Given an irreducible A-module V^+ such that $V^+ \not\cong (V^+)^{\delta} =: V^-$, we strive for a converse of (4.1.4). Let $\{v_1^+, \ldots, v_n^+\}$ be a basis of V^+ and denote by $\{v_1^-, \ldots, v_n^-\}$ the same basis of V^- (as vector space). We claim that $V := V^+ \oplus V^-$ is an irreducible A-supermodule. Let $v_{0,j} := v_j^+ + v_j^-$ and $v_{1,j} := v_j^+ - v_j^-$ for $j \in \{1, \ldots, n\}$. Then $V_i := \langle\, v_{i,j} : j = 1, \ldots, n \,\rangle$ is an n-dimensional subspace of V ($i \in \mathbb{Z}_2$). If $v = \sum_{j=1}^{n} \lambda_j v_{0,j} = \sum_{j=1}^{n} \mu_j v_{1,j} \in V_0 \cap V_1$, then $\mu_j = \lambda_j = -\mu_j = 0$ for all $j \in \{1, \ldots, n\}$. Thus $V = V_0 \oplus V_1$ is an F-superspace. Let $i, j, k \in \mathbb{Z}_2$ and $a_k \in A_k$, and suppose that $v_j^+ a_k = \sum_{l=1}^{n} \lambda_l v_l^+ \in V^+$. Note that

$v_j^- a_k = \sum_{l=1}^n \lambda_l v_l^-$, whereas $v_j^- \cdot a_k = v_j^- a_k^\delta = (-1)^k v_j^- a_k$. As

$$v_{i,j} a_k = (v_j^+ + (-1)^i v_j^-) a_k = v_j^+ a_k + (-1)^{i+k} v_j^- a_k = \sum_{l=1}^n \lambda_l (v_l^+ + (-1)^{i+k} v_l^-) \in V_{i+k},$$

V is an A-supermodule indeed. Finally, suppose that U is a proper non-zero subsupermodule of V. Then $|U|$ is a proper non-zero submodule of $|V|$, that is, either V^+ or V^-. Now the δ_V-invariance of U induces an $|A|$-homomorphism $\iota : U \to U^\delta$, $u \mapsto u^{\delta_V}$. It follows that $V^+ \cong V^-$, a contradiction.

(4.1.6) Proposition. Let A be a superalgebra over F. If V and V^{δ_A} are non-isomorphic irreducible A-modules, then $V \oplus V^{\delta_A}$ is an irreducible A-supermodule of type $\mathtt{Q}$. ∎

We quote two results that need a characterization of *(semi)simple* superalgebras, see [32, (12.2.10), (12.2.11)].

(4.1.7) Theorem. Let A be a superalgebra over F. Suppose that F is algebraically closed.

(1) Assume that $\{V_1, \dots, V_n\}$ is a complete set of pairwise non-isomorphic irreducible A-supermodules such that $V_1, \dots, V_m$ are of type $\mathtt{M}$ and $V_{m+1}, \dots, V_n$ are of type $\mathtt{Q}$. Then $\mathcal{I} := \{|V_1|, \dots, |V_m|, V_{m+1}^\pm, \dots, V_n^\pm\}$ is a complete set of pairwise non-isomorphic irreducible A-modules.
Moreover, the algebra automorphism $\delta := \delta_A$ acts on $\mathcal{I}$ by sending an A-module W to the same vector space with twisted action $w \cdot a := wa^\delta$ for $w \in W$, $a \in A$ such that $|V_i|^\delta \cong |V_i|$ and $(V_j^\pm)^\delta \cong V_j^\mp$ for all $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$.

(2) Suppose that A_1 contains an invertible element s. Given an A_0-module U, we denote by U^s the *twisted* A_0-module that is given by the vector space U with action $u \cdot a := us^{-1}as$ for $u \in U$, $a \in A_0$.
For $i \in \{1, \dots, m\}$, the restriction $|V_i| \downarrow_{A_0} \cong U_i^+ \oplus U_i^-$ is a direct sum of non-isomorphic irreducible A_0-modules such that $(U_i^+)^s \cong U_i^-$; for $j \in \{m+1, \dots, n\}$, $U_j := V_j^+ \downarrow_{A_0} \cong V_j^- \downarrow_{A_0}$ is irreducible. A complete set of pairwise non-isomorphic irreducible A_0-modules is given by $\{U_1^\pm, \dots, U_m^\pm, U_{m+1}, \dots, U_n\}$. ∎

(4.1.8) Remark. Let A be a superalgebra over a field K with $\operatorname{char} K \neq 2$. We call K a *splitting field* for A if for every A-supermodule V of type $\mathtt{M}$ we have $\mathrm{End}_A(V) \cong K$, and for every A-supermodule V of type $\mathtt{Q}$ it holds that $\mathrm{End}_A(V)_i \cong K$ for all $i \in \mathbb{Z}_2$. Suppose that K is a splitting field for A. Clearly, if V is an A-supermodule of type

$\mathtt{M}$, then $|V|$ is an irreducible A-module with $\mathrm{End}_A(|V|) \cong K$. If V is of type $\mathtt{Q}$, then $\dim_K \mathrm{End}_A(V) = 2$ and $|V| = U \oplus U^{\delta_V}$ as in the proof of (4.1.4), where U and U^{δ_V} are irreducible A-submodules of $|V|$. Since both, $\mathrm{End}_A(U)$ and $\mathrm{End}_A(U^{\delta_V})$, are at least one dimensional over K, part (1) of (4.1.2) forces $\mathrm{Hom}_A(U, U^{\delta_V}) = 0$. Therefore, $U \not\cong U^{\delta_V}$. In particular, K is a splitting field for $|A|$. Conversely, let K be a splitting field for $|A|$. Again, $\dim_K \mathrm{End}_A(|V|) = \dim_K \mathrm{End}_A(V) = 1$ if V is of type $\mathtt{M}$. If V is of type $\mathtt{Q}$, then we can deduce $\mathrm{Hom}_A(U, U^{\delta_V}) = 0$ from consideration of $V = U \oplus U^{\delta_V}$ over the algebraic closure of K and application of (4.1.4). Then $\dim_K \mathrm{End}_A(V) = 2$. Since $\mathrm{id}_V \in \mathrm{End}_A(V)_0$ and $J_V \in \mathrm{End}_A(V)_1$, it follows that $\mathrm{End}_A(V)_0 \cong \mathrm{End}_A(V)_1 \cong K$. Thus K is a splitting field for the superalgebra A if and only if K is a splitting field for $|A|$.
The preceding discussion also shows that (4.1.4) holds under the requirement that F is a splitting field for A (instead of being algebraically closed).

4.2 Spin representations

We adhere to our standard setting as introduced in Section 1 for $\tilde{S}_n$ with respect to an odd rational prime $p \mid n$ and p-modular splitting system $(\mathcal{F}, \mathcal{O}, f, \eta)$. Let be $F \in \{\mathcal{F}, f\}$. We may turn $F\tilde{S}_n$ into a superalgebra with respect to the grading

$$(F\tilde{S}_n)_0 := \langle \tilde{A}_n \rangle_F = F\tilde{A}_n \quad \text{and} \quad (F\tilde{S}_n)_1 := \langle \tilde{S}_n \setminus \tilde{A}_n \rangle_F.$$

A sequence $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ that satisfies $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$ and $\sum_{i=1}^{l} \lambda_i = n$ is a *partition* of n. Each λ_i is a *part* and $l(\lambda) = l$ is the *length* of λ. We denote the set of all partitions of n by $\mathcal{P}(n)$.

(4.2.1) Definition. A partition $\lambda \in \mathcal{P}(n)$ is called p-*strict* if $p \mid \lambda_i$ whenever $\lambda_i = \lambda_{i+1}$ for $i \in \{1, \dots, l(\lambda)-1\}$. Let $\mathcal{P}_p(n)$ denote the set of all p-strict partitions of n. Let $\lambda \in \mathcal{P}_p(n)$ and set $\lambda_{l(\lambda)+1} := 0$. If

$$\begin{cases} \lambda_i - \lambda_{i+1} < p & \text{if } p \mid \lambda_i, \\ \lambda_i - \lambda_{i+1} \leqslant p & \text{if } p \nmid \lambda_i \end{cases}$$

for all $i \in \{1, \dots, l(\lambda)\}$, then λ is *restricted* p-strict. We denote by $\mathcal{RP}_p(n)$ the set of all restricted p-strict partitions of n, and set $a(\lambda) := n - |\{ i : p \nmid \lambda_i \}|$ for $\lambda \in \mathcal{P}_p(n)$.

Let $\mathcal{D}(n)$ denote the set of all partitions of n with pairwise distinct parts. Then $\mathcal{D}(n)$ may be interpreted as $\mathcal{P}_0(n)$ or $\mathcal{RP}_\infty(n)$ with $a(\lambda) = n - l(\lambda)$ for $\lambda \in \mathcal{D}(n)$.

We identify a partition λ of n with its *Young diagram*

$$\{(i, j_i) : i \in \{1, \ldots, l(\lambda)\},\ j_i \in \{1, \ldots, \lambda_i\}\}.$$

For a p-strict partition λ, we label its nodes (or boxes) $(i, j_i) \in \lambda$ with *residues* $\operatorname{res}(i, j_i)$ from $I(p) := \{0, 1, \ldots, \ell\}$, where $\ell := (p-1)/2$, concatenating the pattern

$$0, 1, \ldots, \ell - 1, \ell, \ell - 1, \ldots, 1, 0$$

along the i-th row $\{(i, j) : j \in \{1, \ldots, \lambda_i\}\}$ for all $i \in \{1, \ldots, l(\lambda)\}$. The *residue content* of λ is $\operatorname{cont}(\lambda) := (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ where γ_i is the number of nodes of residue $i \in I(p)$. Note that the residues and the labelling scheme differ from their analogues for symmetric groups.

To remove a p-*bar* from $\lambda \in \mathcal{D}(n) \cup \mathcal{P}_p(n)$ means

- remove a part $\lambda_i = p$ from λ, or
- subtract p from a part $\lambda_i > p$ if $\lambda_i - p$ is not a part of λ, or
- remove the parts λ_i and λ_j from λ if $\lambda_i + \lambda_j = p$.

In addition, to remove a p-*bar* from $\lambda \in \mathcal{P}_p(n)$ may also mean

- subtract p from a part $\lambda_i > p$ if $\lambda_i - p$ is a part of λ and $p \mid \lambda_i$.

Removing a p-bar from λ and possibly reordering the remaining parts yields an element of $\mathcal{D}(n-p)$ or $\mathcal{P}_p(n-p)$, respectively. The successive removal of p-bars leads eventually to the p-*bar core* $c_p(\lambda)$ of λ. The p-weight $w_p(\lambda)$ of λ is defined as the number of p-bar removals until $c_p(\lambda)$ is reached. For instance, consider $\lambda = (11, 7^2, 4, 3, 1) \in \mathcal{P}_7(33)$: its Young diagram, labelled with residues, is

0	1	2	3	2	1	0	0	1	2	3
0	1	2	3	2	1	0				
0	1	2	3	2	1	0				
0	1	2	3							
0	1	2								
0										

and $\operatorname{cont}(\lambda) = (10, 9, 9, 5)$. Moreover, $c_7(\lambda) = (4, 1)$ and $w_7(\lambda) = 4$.

By (4.1.6), Schur's original classification of the irreducible spin $\mathcal{F}\tilde{S}_n$-modules (Section X, §41.IX in [67]) may be formulated as follows:

(4.2.2) Theorem (Schur). For every $\lambda \in \mathcal{D}(n)$ there is an irreducible spin $\mathcal{F}\tilde{S}_n$-supermodule $X(\lambda)$ of type

$$\begin{cases} \mathtt{M} & \text{if } a(\lambda) = n - l(\lambda) \text{ is even,} \\ \mathtt{Q} & \text{if } a(\lambda) = n - l(\lambda) \text{ is odd.} \end{cases}$$

Then $\{ X(\lambda) : \lambda \in \mathcal{D}(n) \}$ is a complete set of irreducible and pairwise non-isomorphic spin $\mathcal{F}\tilde{S}_n$-supermodules. ∎

The distribution of ordinary spin modules into p-blocks, see [23], translates as

(4.2.3) Theorem. Let $\lambda, \mu \in \mathcal{D}(n)$. Then $X(\lambda)$ and $X(\mu)$ belong to the same p-block of $\tilde{S}_n$ if and only if $c_p(\lambda) = c_p(\mu)$.

The next theorem is a classification of irreducible spin $f\tilde{S}_n$-supermodules worked out by Brundan and Kleshchev, see [32, 22.3.1]. Let τ denote the superalgebra antiautomorphism of $f\tilde{S}_n$ that is defined by $t \mapsto t^{-1}$ for $t \in \tilde{S}_n$. Recall that $a(\lambda) = n - |\{ i : p \nmid \lambda_i \}|$ for $\lambda \in \mathcal{RP}_p(n)$.

(4.2.4) Theorem (Brundan–Kleshchev). For every $\lambda \in \mathcal{RP}_p(n)$ there is an irreducible spin $f\tilde{S}_n$-supermodule $D(\lambda)$ of type

$$\begin{cases} \mathtt{M} & \text{if } a(\lambda) \text{ is even,} \\ \mathtt{Q} & \text{if } a(\lambda) \text{ is odd.} \end{cases}$$

Then $\{ D(\lambda) : \lambda \in \mathcal{RP}_p(n) \}$ is a complete set of irreducible and pairwise non-isomorphic spin $f\tilde{S}_n$-supermodules. Moreover,

(1) $D(\lambda)$ and $D(\lambda)^\tau$ are isomorphic,

(2) $D(\lambda)$ and $D(\mu)$ belong to the same p-block if and only if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$, and

(3) $D(\lambda)$ is projective if and only if λ is a p-bar core. ∎

(4.2.5) Remark. It is stated in [32, p. 255 eq. (22.7)] that for for $\lambda, \mu \in \mathcal{P}_p(n)$ we have $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ if and only if $c_p(\lambda) = c_p(\mu)$. Hence (4.2.4).(2) resembles the block distribution in the ordinary case (4.2.3). See (4.3.1) below.

Now (4.2.2) and (4.2.4) say that (irreducible) spin modules of $\mathcal{F}\tilde{S}_n$ are parameterized by $\mathcal{D}(n)$, and spin modules of $f\tilde{S}_n$ are parameterized by $\mathcal{RP}_p(n)$. Let $\lambda \in \mathcal{RP}_p(n)$. If $a(\lambda)$ is even, then $|D(\lambda)|$ is an irreducible $f\tilde{S}_n$-module that we denote by $D(\lambda, 0)$. We will denote its Brauer character by $[\![\lambda]\!]$. If $a(\lambda)$ is odd, then $|D(\lambda)|$ decomposes as a direct sum of irreducible $f\tilde{S}_n$-modules $D(\lambda, +)$ and $D(\lambda, -)$ whose Brauer characters will be denoted by $[\![\lambda, +]\!]$ and $[\![\lambda, -]\!]$. We use an analogous notation for spin modules over $\mathcal{F}$: for $\lambda \in \mathcal{D}(n)$ we interchange 'D' with 'X' and use double angle brackets $\langle\!\langle\cdot\rangle\!\rangle$ to denote the corresponding ordinary characters.

For $F \in \{\mathcal{F}, f\}$, let M be an $F\tilde{S}_n$-supermodule and $\delta := \delta_{F\tilde{S}_n}$. It is easy to see that the twisted $F\tilde{S}_n$-supermodule M^{δ} is exactly the associate $F\tilde{S}_n$-module $M^{a} = M \otimes \mathfrak{s}$, where $\mathfrak{s} = \mathfrak{s}_{\tilde{S}_n}$ denotes the sign module of $F\tilde{S}_n$. In particular, if $M(\lambda)$ is an $F\tilde{S}_n$-supermodule of type M, then $M(\lambda, 0)$ is self-associate, that is, $M(\lambda, 0) \cong M(\lambda, 0) \otimes \mathfrak{s}$; if $M(\lambda)$ is of type Q, then $M(\lambda, -) \cong M(\lambda, +) \otimes \mathfrak{s}$.

The classification of spin $F\tilde{A}_n$-modules also follows from the theorems above and (4.1.7): if $a(\lambda)$ is even, then there are non-isomorphic spin $F\tilde{A}_n$-modules $N(\lambda, +)$ and $N(\lambda, -)$ such that $M(\lambda, 0) \downarrow_{F\tilde{A}_n} \cong N(\lambda, +) \oplus N(\lambda, -)$. Otherwise, that is if $a(\lambda)$ is odd, then $N(\lambda, 0) := M(\lambda, +)\downarrow_{F\tilde{A}_n} \cong M(\lambda, -)\downarrow_{F\tilde{A}_n}$ is a spin $F\tilde{A}_n$-module.

(4.2.6) Corollary. A complete set of irreducible and pairwise non-isomorphic spin $\mathcal{F}\tilde{S}_n$-modules is given by

$$\{\, X(\lambda, 0) : \lambda \in \mathcal{D}(n), a(\lambda)\,\text{even}\,\} \cup \{\, X(\lambda, +), X(\lambda, -) : \lambda \in \mathcal{D}(n), a(\lambda)\,\text{odd}\,\},$$

and a complete set of irreducible and pairwise non-isomorphic spin $f\tilde{S}_n$-modules is

$$\{\, D(\lambda, 0) : \lambda \in \mathcal{RP}_p(n), a(\lambda)\,\text{even}\,\} \cup \{\, D(\lambda, +), D(\lambda, -) : \lambda \in \mathcal{RP}_p(n), a(\lambda)\,\text{odd}\,\}.$$

Moreover,

$$\{\, Y(\lambda, +), Y(\lambda, -) : \lambda \in \mathcal{D}(n), a(\lambda)\,\text{even}\,\} \cup \{\, Y(\lambda, 0) : \lambda \in \mathcal{D}(n), a(\lambda)\,\text{odd}\,\}$$

is a complete set of irreducible and pairwise non-isomorphic spin $\mathcal{F}\tilde{A}_n$-modules, and

$$\{\, E(\lambda, +), E(\lambda, -) : \lambda \in \mathcal{RP}_p(n), a(\lambda)\,\text{even}\,\} \cup \{\, E(\lambda, 0) : \lambda \in \mathcal{RP}_p(n), a(\lambda)\,\text{odd}\,\}$$

is a complete set of irreducible and pairwise non-isomorphic spin $f\tilde{A}_n$-modules. ∎

Branching rules

Consider $\lambda \in \mathcal{D}(n)$ as a ∞-strict partition and label its Young diagram with residues from $I(\infty) := \{0, 1, 2, \ldots\}$. Then we say that the node $A \in \lambda$ of residue i is i-*removable* if $\lambda_A := \lambda \backslash \{A\} \in \mathcal{D}(n-1)$. A node $B \notin \lambda$ of residue i is i-*addable* if $\lambda^B := \lambda \cup \{B\} \in \mathcal{D}(n+1)$. Note that for every $i \in I(\infty)$ there is either no i-removable/addable node of λ, or exactly one.

(4.2.7) Theorem. For $\lambda \in \mathcal{D}(n)$, define $I := \{ i \in I(\infty) : \lambda \text{ has an } i\text{-removable node } A_i \}$ and $J := \{ i \in I(\infty) : \lambda \text{ has an } i\text{-addable node } B_i \}$. Then

$$X(\lambda){\downarrow_{\tilde{S}_{n-1}}} \cong \bigoplus_{i \in I} m_i X(\lambda_{A_i}) \quad \text{and} \quad X(\lambda){\uparrow^{\tilde{S}_{n+1}}} \cong \bigoplus_{i \in J} m_i X(\lambda^{B_i}),$$

where $m_i = 1$ if $a(\lambda)$ is even or $i = 0$, and $m_i = 2$ otherwise. ∎

Now consider $\lambda \in \mathcal{P}_p(n)$ and a fixed residue $i \in I(p)$. A node $A = (r, s) \in \lambda$ is i-*removable* if $\operatorname{res} A = i$ and $\lambda_A := \lambda \setminus \{A\} \in \mathcal{P}_p(n-1)$. Also, A is 0-removable if $B := (r, s+1) \in \lambda$, $\operatorname{res} A = \operatorname{res} B = 0$, $\lambda_B \in \mathcal{P}_p(n-1)$, and $\lambda_{B,A} := \lambda \setminus \{A, B\} \in \mathcal{P}_p(n-2)$ (in which case B is 0-removable too). Analogously, a node $A = (r, s) \notin \lambda$ is i-*addable* if $\operatorname{res} A = i$ and $\lambda^A := \lambda \cup \{A\} \in \mathcal{P}_p(n+1)$. In addition, A is 0-addable if $B := (r, s-1) \notin \lambda$, $\operatorname{res} A = \operatorname{res} B = 0$, $\lambda^B \in \mathcal{P}_p(n+1)$, and $\lambda^{B,A} := \lambda \cup \{A, B\} \in \mathcal{P}_p(n+2)$.

Label the i-addable nodes by $+$ and the i-removable nodes by $-$. The sequence of signs $\epsilon \in \{+, -\}$ along the rim from bottom left to top right is called the i-*signature* of λ. The successive removal of all subsequences $+-$ from the i-signature eventually gives the *reduced* i-*signature* of λ. A node corresponding to a sign ϵ in the reduced i-signature is called i-*normal* if $\epsilon = -$, and i-*conormal* if $\epsilon = +$. The i-normal node corresponding to the rightmost $-$ in the reduced i-signature of λ is i-*good*, the i-conormal node corresponding to the leftmost $+$ is i-*cogood*. Finally, let $\varepsilon_i(\lambda)$ be the number of i-normal nodes of λ, and let $\varphi_i(\lambda)$ be the number of i-conormal nodes of λ.

The restriction and induction of spin supermodules between $f\tilde{S}_n$ and $f\tilde{S}_{n-1}$ is described by the next two theorems, see (22.3.4) and (22.3.5) in [32].

(4.2.8) Theorem. Let $\lambda \in \mathcal{RP}_p(n)$ and $I := \{ i \in I(p) : \lambda \text{ has an } i\text{-good node } A_i \}$. Then there exist non-zero self-dual indecomposable $f\tilde{S}_{n-1}$-supermodules $e_i D(\lambda)$, unique up to isomorphism, such that

$$D(\lambda){\downarrow_{\tilde{S}_{n-1}}} \cong \bigoplus_{i \in I} m_i e_i D(\lambda)$$

where $m_i = 1$ if $a(\lambda)$ is even or $i = 0$, and $m_i = 2$ otherwise. For every $i \in I$:

(1) the head and the socle of $e_i D(\lambda)$ are isomorphic to $D(\lambda_{A_i})$,

(2) the multiplicity of $D(\lambda_{A_i})$ in $e_i D(\lambda)$ is $\varepsilon_i(\lambda)$,

(3) $\varepsilon_i(\lambda_{A_i}) = \varepsilon_i(\lambda) - 1$,

(4) $\varepsilon_i(\mu) < \varepsilon_i(\lambda) - 1$ for all other composition factors $D(\mu)$ of $e_i D(\lambda)$,

(5) $\dim_f \mathrm{End}_{f\tilde{S}_{n-1}}(e_i D(\lambda)) = \varepsilon_i(\lambda) \cdot \dim_f \mathrm{End}_{f\tilde{S}_{n-1}}(D(\lambda_{A_i}))$,

(6) $\mathrm{Hom}_{f\tilde{S}_{n-1}}(e_i D(\lambda), e_i D(\mu)) = 0$ for all $\mu \in \mathcal{RP}_p(n)$ with $\mu \neq \lambda$,

(7) $e_i D(\lambda)$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$,

(8) $D(\lambda){\downarrow}_{\tilde{S}_{n-1}}$ is completely reducible if and only if $\varepsilon_i(\lambda) \leqslant 1$ for all $i \in I(p)$. ∎

(4.2.9) Theorem. Let $\lambda \in \mathcal{RP}_p(n)$ and $I := \{i \in I(p) : \lambda \text{ has an } i\text{-cogood node } B_i\}$. Then there exist non-zero self-dual indecomposable $f\tilde{S}_{n+1}$-supermodules $f_i D(\lambda)$, unique up to isomorphism, such that

$$D(\lambda){\uparrow}^{\tilde{S}_{n+1}} \cong \bigoplus_{i \in I} m_i f_i D(\lambda)$$

where $m_i = 1$ if $a(\lambda)$ is even or $i = 0$, and $m_i = 2$ otherwise. For every $i \in I$:

(1) the head and the socle of $f_i D(\lambda)$ are isomorphic to $D(\lambda^{B_i})$,

(2) the multiplicity of $D(\lambda^{B_i})$ in $f_i D(\lambda)$ is $\varphi_i(\lambda)$,

(3) $\varphi_i(\lambda^{B_i}) = \varphi_i(\lambda) - 1$,

(4) $\varphi_i(\mu) < \varphi_i(\lambda) - 1$ for all other composition factors $D(\mu)$ of $f_i D(\lambda)$,

(5) $\dim_f \mathrm{End}_{f\tilde{S}_{n+1}}(f_i D(\lambda)) = \varphi_i(\lambda) \cdot \dim_f \mathrm{End}_{f\tilde{S}_{n+1}}(D(\lambda^{B_i}))$,

(6) $\mathrm{Hom}_{f\tilde{S}_{n+1}}(f_i D(\lambda), f_i D(\mu)) = 0$ for all $\mu \in \mathcal{RP}_p(n)$ with $\mu \neq \lambda$,

(7) $f_i D(\lambda)$ is irreducible if and only if $\varphi_i(\lambda) = 1$,

(8) $D(\lambda){\uparrow}^{\tilde{S}_{n+1}}$ is completely reducible if and only if $\varphi_i(\lambda) \leqslant 1$ for all $i \in I(p)$. ∎

4.3 Labels for characters I

Denote by $\mathcal{S}(n)$ the set of Brauer characters afforded by the spin $f\tilde{S}_n$-supermodules $D(\lambda)$, $\lambda \in \mathcal{RP}_p(n)$. Provided we know the set $\mathcal{I}(\tilde{S}_n) \subset \mathrm{IBr}(\tilde{S}_n)$ of the faithful irreducible Brauer characters of $\tilde{S}_n$ (with respect to our choice of p-modular system),

$$\mathcal{S}(n) = \{\, \phi : \phi \in \mathcal{I}(\tilde{S}_n),\ \phi = \phi^a \,\} \cup \{\, \phi + \phi^a : \phi \in \mathcal{I}(\tilde{S}_n),\ \phi \neq \phi^a \,\}$$

is the set of spin "supercharacters" of $\tilde{S}_n$. For $\phi \in \mathcal{S}(n)$, define the *type* of ϕ as the type of its corresponding supermodule, and set

$$a(\phi) := \begin{cases} 0 & \text{if } \phi \text{ is of type } \mathtt{M}, \\ 1 & \text{if } \phi \text{ is of type } \mathtt{Q}. \end{cases}$$

By means of (4.2.8), we try to match characters from $\mathcal{S}(n)$ with labels from $\mathcal{RP}_p(n)$ by induction with respect to n. Assume that

$$\mathcal{S}(n-1) = \{\, [\![\mu]\!] : \mu \in \mathcal{RP}_p(n-1) \,\}$$

and decompose $\mathcal{S}(n){\downarrow_{\tilde{S}_{n-1}}}$ in terms of $\mathcal{S}(n-1)$, say if $\phi \in \mathcal{S}(n)$, then

$$\phi{\downarrow_{\tilde{S}_{n-1}}} = \sum_{\mu \in \mathcal{RP}_p(n-1)} r_{\phi\mu} [\![\mu]\!].$$

Fix a label $\lambda \in \mathcal{RP}_p(n)$ and determine its "good residues" with corresponding nodes,

$$I := \{\, i \in I(p) : \lambda \text{ has an } i\text{-good node } A_i \,\}.$$

Suppose that $\phi = [\![\lambda]\!]$. If $\mu = \lambda_{A_i}$ for some $i \in I$, then $D(\mu)$ is involved in $e_i D(\lambda)$ with multiplicity $\varepsilon_i(\lambda)$. Thus $m_i \varepsilon_i(\lambda) \leqslant r_{\phi\mu}$ with m_i as in (4.2.8). In addition, if the inequality is strict, then there exists some $j \in I \setminus \{i\}$ such that $D(\mu)$ is involved in $e_j D(\lambda)$. This forces $\varepsilon_j(\mu) < \varepsilon_j(\lambda) - 1$. Also, if $\mu \neq \lambda_{A_i}$ for all $i \in I$ and $r_{\phi\mu} > 0$, then there exists some $j \in I$ such that $D(\mu)$ is involved in $e_j D(\lambda)$ and $\varepsilon_j(\mu) < \varepsilon_j(\lambda) - 1$. Hence we have the following necessary conditions

(N1) ϕ and $D(\lambda)$ must have the same type, that is $a(\lambda) \equiv a(\phi) \bmod 2$, (N2) $m_i\varepsilon_i(\lambda) \leqslant r_{\phi\lambda_{A_i}}$ for all $i \in I$, where

$$\begin{cases} m_i = 1 & \text{if } a(\lambda) \text{ is even or } i = 0, \\ m_i = 2 & \text{otherwise,} \end{cases}$$

(N3) if $\varepsilon_i(\lambda) < r_{\phi\lambda_{A_i}}$, then $\varepsilon_j(\lambda_{A_i}) < \varepsilon_j(\lambda) - 1$ for some $j \in I \setminus \{i\}$,

(N4) if $\mu \in \mathcal{RP}_p(n-1) \setminus \{\lambda_{A_i} : i \in I\}$ with $r_{\phi\mu} > 0$ then $\varepsilon_j(\mu) < \varepsilon_j(\lambda) - 1$ for some $j \in I$.

We have implemented these conditions in an algorithm in GAP that runs through $\mathcal{RP}_p(n)$ in reverse lexicographical order ($\trianglerighteq$). In each step, the prospective $\phi \in \mathcal{S}(n)$ are filtered subject to the conditions (N1)-(N4) on $[r_{\phi\mu}]_{\mu\in\mathcal{RP}_p(n-1)}$. Whenever there is only one candidate ϕ for a given label λ, we assign $\phi = [\![\lambda]\!]$ and remove ϕ from $\mathcal{S}(n)$. This method turned out to be sufficient for all $n \leqslant 18$ with $p \in \{3, 5, 7\}$.
Once we know the label of the supercharacter $\phi \in \mathcal{S}(n)$, we may assign the corresponding label(s) to its constituent(s) from $\mathrm{IBr}(\tilde{S}_n)$. Our *choice* of labels is as follows: if the supercharacter ϕ afforded by $D(\lambda)$ is of type `Q` with constituents $\phi^+, \phi^- \in \mathrm{IBr}(\tilde{S}_n)$ such that $\phi^+ > \phi^-$ with respect to the order $>$ on class functions in GAP, then $\phi^+ = [\![\lambda, +]\!]$ and $\phi^- = [\![\lambda, -]\!]$ are the Brauer characters afforded by $D(\lambda, +)$ and $D(\lambda, -)$, respectively. If ϕ is of type `M`, then $\phi = [\![\lambda]\!]$ is just the Brauer character of $D(\lambda, 0)$. Moreover, the labels for elements of $\mathrm{IBr}(\tilde{A}_n)$ are given implicitly by the labels from $\mathrm{IBr}(\tilde{S}_n)$. We use the analogous choice of signs $\pm$ to distinguish conjugate Brauer characters of $\tilde{A}_n$.

(4.3.1) Remark. Our approach to matching characters with their combinatorial labels is rather global than local: although we separate characters and labels according to their type (see (N1)), we do not divide them into blocks. That is because it is not entirely clear that the p-bar core of an ordinary spin block coincides with the p-bar core of a modular spin block (to the best knowledge of the author), see (4.2.5). Indeed, it happens in all examples known to the author. If this is true in general, then [10, Theorem 1.2(ii)] can be used to achieve the matching blockwise. However, our approach seems to be of a different nature since the result of [10] depends on *shadow nodes*, in contrast to the good nodes that determine the branching here. (See, for instance, Example 22.1.1 in [32]: the shadow node of $\lambda = (16, 11, 10^2, 9, 5, 1)$ is $(6, 5)$, whereas the 0-good node of λ is $(1, 16)$.)

5 More on $\tilde{S}_n$

Recall that $\tilde{S}_n$ is generated by elements $z, t_1, \dots, t_{n-1}$ subject to the relations

$$\begin{aligned} z^2 &= 1, \\ t_i^2 &= z \quad \text{for all} \quad i \in \{1, \dots, n-1\}, \\ (t_i t_{i+1})^3 &= z \quad \text{for all} \quad i \in \{1, \dots, n-2\}, \\ (t_i t_j)^2 &= z \quad \text{for all} \quad i, j \in \{1, \dots, n-1\} \quad \text{such that} \quad |i-j| > 1. \end{aligned}$$

Then $\tilde{A}_n = A_n^{\pi^{-1}}$ is the preimage of A_n under the natural epimorphism $\pi : \tilde{S}_n \to S_n$. We may choose $t_i^\pi = (i, i+1)$ here.

5.1 Conjugacy in $\tilde{S}_n$

Suppose that we are given elements $x, y \in \tilde{S}_n$ such that x^π and y^π have disjoint sets of non-fixed points. Then there exists an element $g \in \tilde{S}_{n-1}$ such that $(x^\pi)^g$ acts on $\{1, \dots, k\}$ and $(y^\pi)^g$ acts on $\{k+1, \dots, n\}$ for some $k \leqslant n$. Taking $\tilde{g} \in \tilde{S}_n$ such that $\tilde{g}^\pi = g$, we have $x^{\tilde{g}} \in \langle t_1, \dots, t_{k-1} \rangle$ and $y^{\tilde{g}} \in \langle t_{k+1}, \dots, t_{n-1} \rangle$, say $x^{\tilde{g}} = t_{i_1} \cdots t_{i_r}$ and $y^{\tilde{g}} = t_{j_1} \cdots t_{j_s}$ for suitable indices $i_1, \dots, i_r \in \{1, \dots, k-1\}$ and $j_1, \dots, j_s \in \{k+1, \dots, n-1\}$. Thus $(xy)^{\tilde{g}} = x^{\tilde{g}} y^{\tilde{g}} = z^{rs} y^{\tilde{g}} x^{\tilde{g}} = z^{rs} (yx)^{\tilde{g}}$ as $t_{i_a} t_{j_b} = z t_{j_b} t_{i_a}$ for all $a \in \{1, \dots, r\}$, $b \in \{1, \dots, s\}$.

(5.1.1) Proposition. Let $x, y \in \tilde{S}_n$ such that x^π and y^π act on disjoint subsets of $\{1, \dots, n\}$. If $x = \prod_{i \in I} t_i$ and $y = \prod_{j \in J} t_j$ for some index multisets I and J, then $xy = z^{|I| \cdot |J|} yx$. ∎

An element $x = \prod_{i \in I} t_i \in \tilde{S}_n$ is called *even* if $x \in \tilde{A}_n$, and *odd* otherwise. As the sign of x^π is $(-1)^{|I|}$, in (5.1.1) we have $xy = yx$ unless both x and y are odd, in which case $xy = zyx$.

We call a sequence $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ that satisfies $\sum_{i=1}^{l} \lambda_i = n$ a *composition* of n. Each λ_i is a *part* and $l(\lambda) = l$ is the *length* of λ. Denote by $\mathcal{C}(n)$ the set of all compositions of n. If $\lambda \in \mathcal{C}(n)$, then reordering its parts λ_i with respect to $\geqslant$ yields a partition $\lambda_{\geqslant} \in \mathcal{P}(n)$. A partition is *even* if it has an even number of even parts, and *odd* otherwise. Let $\mathcal{P}^+(n)$ and $\mathcal{P}^-(n)$ denote the subsets of $\mathcal{P}(n)$ containing all even and all odd partitions, respectively. In addition, we have

$$\begin{aligned}\mathcal{O}(n) &:= \{\lambda \in \mathcal{P}(n) : \lambda_i \text{ odd for all } i\},\\ \mathcal{D}(n) &:= \{\lambda \in \mathcal{P}(n) : \lambda_i \neq \lambda_j \text{ for all } i \neq j\},\\ \mathcal{D}^{\pm}(n) &:= \mathcal{D}(n) \cap \mathcal{P}^{\pm}(n).\end{aligned}$$

(5.1.2) Definition. Let $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{C}(n)$ be a composition of n. We set $\ell_1 := 0$, $\ell_i := \lambda_1 + \cdots + \lambda_{i-1}$ for $i \geqslant 2$, and $t(\lambda_i, \ell_i) := t_{\ell_i+1} t_{\ell_i+2} \cdots t_{\ell_i+\lambda_i-1}$ for $i \in \{1, \dots, l-1\}$, where $t(\lambda_i, \ell_i) = 1$ if $\lambda_i = 1$. The *standard representative of type* λ is defined as

$$t_\lambda := \prod_{i=1}^{l-1} t(\lambda_i, \ell_i).$$

(5.1.3) Example. Let $\lambda = (4, 1, 3, 7) \in \mathcal{C}(15)$. Then $(\ell_1, \ell_2, \ell_3, \ell_4) = (0, 4, 5, 8)$ and

$$t_\lambda = t_1 t_2 t_3 \cdot t_6 t_7 \cdot t_9 t_{10} t_{11} t_{12} t_{13} t_{14}.$$

Moreover, $\Lambda := \lambda_{\geqslant} = (7, 4, 3, 1) \in \mathcal{P}(15)$ and

$$t_\Lambda = t_1 t_2 t_3 t_4 t_5 t_6 \cdot t_8 t_9 t_{10} \cdot t_{12} t_{13}.$$

Note that t_λ^π and t_Λ^π are conjugate in S_{15}. Are t_λ and t_Λ conjugate in $\tilde{S}_{15}$?

The conjugacy classes of S_n are naturally parameterized by the cycle types of their elements, and each cycle type corresponds to a partition of n. Therefore a conjugacy class C of S_n is characterized by its *type* $\lambda \in \mathcal{P}(n)$. Assume for a moment that $C \subset A_n$. Then C is also a conjugacy class of A_n if and only if $\lambda \notin \mathcal{O}(n) \cap \mathcal{D}^+(n)$. Otherwise, C *splits* as the union of two distinct A_n-classes of the same size, C^+ of type λ^+ and C^- of type λ^-. (Only in the former case does the centralizer of an element $x \in C$ in S_n contain an odd element.) Sometimes it is convenient to neglect the signs and simply speak of an A_n-class of type λ, keeping in mind that this may mean one of the two classes of type $\lambda^\pm$. Furthermore, $\tilde{C} := C^{\pi^{-1}} \subset \tilde{S}_n$ is either a conjugacy class of $\tilde{S}_n$ (of type λ), or it splits as the union of two distinct $\tilde{S}_n$-classes $\tilde{C}_1$ and $\tilde{C}_2 = z\tilde{C}_1$ (both of type λ). If $\tilde{C}$ is a conjugacy class of type λ, then we define t_λ to be the *standard representative* of $\tilde{C}$. If $\tilde{C}$ splits, then we choose t_λ as the standard representative of the *first class* $\tilde{C}_1$ and zt_λ as the standard representative of the *second class* $\tilde{C}_2$. Similar notions apply for $\tilde{A}_n$. The conjugacy classes whose preimages split are characterized as follows:

(5.1.4) Theorem ([67, Section II, §7, §9]). Let λ be a partition of n.

(1) Let C be a S_n-conjugacy class of type λ. Then $\tilde{C}$ splits in $\tilde{S}_n$ if and only if $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$.

(2) Let C be an A_n-conjugacy class of type λ. Then $\tilde{C}$ splits in $\tilde{A}_n$ if and only if $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^+(n)$. ∎

Note that, if $\lambda \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$, then there are four distinct $\tilde{A}_n$-classes $\tilde{C}_1^+$, $\tilde{C}_2^+$, $\tilde{C}_1^-$, $\tilde{C}_2^-$ with standard representatives $t_{\lambda^+} := t_\lambda$, $zt_{\lambda+}$, t_{λ^-}, $zt_{\lambda-}$, respectively.

In Section 1 we have seen that $GF(p)$ is a splitting field for S_n. In general this is false for $\tilde{S}_n$, $\hat{S}_n$, and $\tilde{A}_n$. However, the situation is not too bad:

(5.1.5) Corollary. $GF(p^2)$ is a splitting field for both $\tilde{S}_n$ and $\tilde{A}_n$.

Proof. Let $q := p^2$ and consider $\tilde{S}_n$ first. By (1.9), we have to check that $x^q \sim_{\tilde{S}_n} x$ for a complete set of representatives x of the conjugacy classes of $\tilde{S}_n$. Clearly, we have $t_\lambda^p \sim_{\tilde{S}_n} t_\lambda$ if $\lambda \in \mathcal{P}(n) \setminus (\mathcal{O}(n) \cup \mathcal{D}^-(n))$. Suppose that $t_\lambda^p \not\sim_{\tilde{S}_n} t_\lambda$ for $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$. Then $t_\lambda^p \sim_{\tilde{S}_n} zt_\lambda$ and $t_\lambda^q \sim_{\tilde{S}_n} t_\lambda$. As $t_{\lambda^\pm}$ and $zt_{\lambda^\pm}$ for $\lambda \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$ have different orders, the claim follows for $\tilde{A}_n$ as well. ∎

Note that $GF(p^2)$ contains a primitive 4th root of unity if $p \neq 2$, thus $GF(p^2)$ is a splitting field for $\hat{S}_n$ too.

Let $\lambda \in \mathcal{C}(n)$ and $\Lambda := \lambda_{\geqslant}$. As both t_λ and t_Λ project to elements of cycle type Λ in S_n, clearly t_λ^π and t_Λ^π are conjugate in S_n. Suppose that $\Lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$, so that t_Λ and zt_Λ are not conjugate in $\tilde{S}_n$. A natural question is whether t_λ and t_Λ, or t_λ and zt_Λ, are conjugate in $\tilde{S}_n$. If $\Lambda \in \mathcal{O}(n)$, then we can easily give the answer by consideration of the elements' orders, but if $\Lambda \in \mathcal{D}^-(n)$, then this approach fails as t_Λ and zt_Λ have equal (even) order. Practical relevance of this question also arises from the study of class fusions of certain subgroups of $\tilde{S}_n$ or $\tilde{A}_n$, see 5.4. It turns out that the answer is encoded in the process of "shuffling" from λ to Λ, which corresponds to moving gradually from t_λ^π towards t_Λ^π by conjugation.

For instance, in (5.1.3) we can shuffle

$$\lambda = (4,1,3,7) \to (4,1,7,3) \to (4,7,1,3) \to (7,4,1,3) \to (7,4,3,1) = \Lambda$$

in correspondence with $(t^\pi_\lambda)^{\alpha\beta\gamma\delta} = t^\pi_\Lambda$ where $\alpha := (6,13)(7,14)(8,15)$, $\beta := (5,12)$, $\gamma := (1,8)(2,9)(3,10)(4,11)$, and $\delta := (12,15)$ are products of suitable transpositions:

$$t^\pi_\lambda \xrightarrow{\alpha} t^\pi_{(4,1,7,3)} \xrightarrow{\beta} t^\pi_{(4,7,1,3)} \xrightarrow{\gamma} t^\pi_{(7,4,1,3)} \xrightarrow{\delta} t^\pi_\Lambda.$$

(5.1.6) Proposition. Suppose that $\lambda \in \mathcal{C}(n)$ with $\Lambda := \lambda_\geqslant \in \mathcal{P}(n)$. We write $(\lambda_i, \lambda_j) \subseteq \lambda$ if λ_i and λ_j are parts of λ such that $i < j$. Then t_λ and t_Λ are conjugate in $\tilde{S}_n$ except possibly when $\Lambda \in \mathcal{D}^-(n)$. In this case, t_λ is conjugate to $z^s t_\Lambda$ where

$$s := |\{ (\lambda_i, \lambda_j) \subseteq \lambda : \lambda_i < \lambda_j \text{ and } \lambda_i \equiv \lambda_j \bmod 2 \}|.$$

We devote the rest of this subsection to the proof of (5.1.6).

(5.1.7) Definition. For $i, j \in \{1, \ldots, n\}$ with $j \geqslant i+1$, let

$$t_{i,j} := t_i t_{i+1} \cdots t_{j-2} t_{j-1} t_{j-2} \cdots t_{i+1} t_i,$$
$$t_{j,i} := t_{j-1} t_{j-2} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-2} t_{j-1}.$$

Note that $t_{i,i+1} = t_i = t_{i+1,i}$ and $t^\pi_{i,j} = (i,j)$. Moreover, $t^{-1}_{i,j} = z t_{i,j}$ and $t^{-1}_{j,i} = z t_{j,i}$.

(5.1.8) Lemma. Let $i, j \in \{1, \ldots, n\}$ be such that $j \geqslant i+2$. Then the following hold:

(1) $t_{i,j} = t_{j,i}$,

(2) $t_i^{t_{i,j}} = t_{i+1,j}$,

(3) $t_j^{t_{i,j}} = z t_{i,j+1}$,

(4) $t_{j-1}^{t_{i,j}} = t_{i,j-1}$,

(5) $t_{i,j}^{t_i} = z t_{i+1,j}$,

(6) $t_{i,j}^{t_{i,j-1}} = z t_{j-1}$,

(7) $t_{i,j}^{t_{i+1,j}} = z t_i$,

(8) $t_{i,j-1}^{t_{i,j}} = t_{j-1}$.

Proof. (1) As $t_{i,i+1} = t_{i+1,i}$, we may assume that $t_{i,j-1} = t_{j-1,i}$. Thus

$$\begin{aligned} t_{j,i} &= t_{j-1} t_{j-1,i} t_{j-1} = t_{j-1} t_{i,j-1} t_{j-1} = t_{j-1} t_i \cdots t_{j-3} t_{j-2} t_{j-3} \cdots t_i t_{j-1} \\ &= t_i \cdots t_{j-3} t_{j-1} t_{j-2} t_{j-1} t_{j-3} \cdots t_i = t_i \cdots t_{j-2} t_{j-1} t_{j-2} \cdots t_i = t_{i,j}. \end{aligned}$$

(2) We have $t_i^{t_{i,i+2}} = zt_it_{i+1}t_i \cdot t_i \cdot t_it_{i+1}t_i = t_it_{i+1}t_it_{i+1}t_i = t_{i+1} = t_{i+1,i+2}$, so let $j > i+2$ and assume that the claim holds for $j-1$. Then

$$t_i^{t_{i,j}} = t_i^{t_{j,i}} = t_i^{t_{j-1}t_{j-1,i}t_{j-1}} = zt_i^{t_{i,j-1}t_{j-1}} = zt_{i+1,j-1}^{t_{j-1}} = t_{j,i+1} = t_{i+1,j}.$$

(3) As $t_{i+2}^{t_{i,i+2}} = zt_it_{i+1}t_it_{i+2}t_it_{i+1}t_i = zt_{i,i+3}$ and $t_jt_{j-1,i} = zt_{j-1,i}t_j$, we get

$$\begin{aligned} t_j^{t_{i,j}} &= t_j^{t_{j,i}} = t_j^{t_{j-1}t_{j-1,i}t_{j-1}} = t_{j-1}^{t_jt_{j-1,i}t_{j-1}} = t_{j-1}^{t_{j-1,i}t_jt_{j-1}} \\ &= zt_{i,j}^{t_jt_{j-1}} = (t_jt_{j,i}t_j)^{t_{j-1}} = t_{j+1,i}^{t_{j-1}} = zt_{i,j+1}. \end{aligned}$$

(4) Using parts (1) and (3), it is straightforward to check

$$t_{j-1}^{t_{i,j}} = t_{j-1}^{t_{j-1}t_{j-1,i}t_{j-1}} = t_{j-1}^{t_{i,j-1}t_{j-1}} = zt_{i,j}^{t_{j-1}} = t_{j-1}t_{j,i}t_{j-1} = t_{j-1,i} = t_{i,j-1}.$$

(5) Clearly, we have $t_{i,j}^{t_i} = zt_it_{i,j}t_i = zt_{i+1,j}$.

(6) As $t_{i,i+1} = t_i$, we have $t_{i,i+2}^{t_{i,i+1}} = zt_i^2t_{i+1}t_i^2 = zt_{i+1}$, and if $j > i+2$, then

$$\begin{aligned} t_{i,j}^{t_{i,j-1}} &= (zt_i\cdots t_{j-3}t_{j-2}t_{j-3}\cdots t_i)(t_i\cdots t_{j-2}t_{j-1}t_{j-2}\cdots t_i)(t_i\cdots t_{j-3}t_{j-2}t_{j-3}\cdots t_i) \\ &= zt_it_{i+1}\cdots t_{j-3}t_{j-1}t_{j-3}\cdots t_{i+1}t_i = z^{1+(j-2-i)}t_{j-1}t_it_{i+1}\cdots t_{j-3}t_{j-3}\cdots t_i \\ &= z^{1+2(j-2-i)}t_{j-1} = zt_{j-1}. \end{aligned}$$

(7) We have $t_{i,i+2}^{t_{i+1,i+2}} = zt_{i+1}t_it_{i+1}t_it_{i+1} = zt_i$, and if $j > i+2$, then

$$\begin{aligned} t_{i,j}^{t_{i+1,j}} &= zt_{i+1,j}t_{i,j}t_{i+1,j} = zt_{j,i+1}t_{j,i}t_{j,i+1} = zt_{j-1}\cdots t_{i+2}t_it_{i+2}\cdots t_{j-1} \\ &= z^{1+(j-i-2)}t_it_{j-2}\cdots t_{i+2}t_{i+2}\cdots t_{j-1} = z^{1+2(j-i-2)}t_i = zt_i. \end{aligned}$$

(8) By parts (1) and (6), we have

$$t_{i,j-1}^{t_{i,j}} = t_{i,j-1}^{t_{j,i}} = t_{i,j-1}^{t_{j-1}t_{j-1,i}t_{j-1}} = zt_{i,j}^{t_{j-1,i}t_{j-1}} = zt_{i,j}^{t_{i,j-1}t_{j-1}} = t_{j-1}^{t_{j-1}} = t_{j-1}.$$ ∎

(5.1.9) Lemma. Let $i, j \in \{2, \ldots, n\}$ be such that $i < j$ and set $c_i := t_{1,j+1}t_{2,j+2}\cdots t_{i,j+i}$. Then

(1) $t_{(i)}^{c_i} = t_{j+1}t_{j+2}\cdots t_{j+i-1}$ for $(i) \in \mathcal{C}(i)$, and

(2) $(t_jt_{j+1}\cdots t_{j+i-1})^{c_i} = z^{i-1}t_{1,j}t_1t_2\cdots t_{i-1} = t_1t_2\cdots t_{i-1}t_{i,j}$.

Proof. (1) If $i = 2$, then parts (2) and (8) of (5.1.8) give

$$t_{(2)}^{c_2} = t_1^{t_{1,j+1}t_{2,j+2}} = t_{2,j+1}^{t_{2,j+2}} = t_{j+1}.$$

Hence we may assume the claim holds for $i - 1 \geqslant 2$. Using (5.1.1),

$$\begin{aligned} t_{(i)}^{c_i} &= (t_1 \cdots t_{i-2})^{c_{i-1}t_{i,j+i}} t_{i-1}^{c_i} = (t_{j+1} \cdots t_{j+i-2})^{t_{i,j+i}} t_{i-1}^{c_{i-2}t_{i-1,j+i-1}t_{i,j+i}} \\ &= z^{i-2} t_{j+1} \cdots t_{j+i-2} z^{i-2} t_{i-1}^{t_{i-1,j+i-1}t_{i,j+i}} = t_{j+1} \cdots t_{j+i-2} t_{i,j+i-1}^{t_{i,j+i}} \\ &= t_{j+1} \cdots t_{j+i-2} t_{j+i-1}. \end{aligned}$$

(2) By parts (3), (4) and (7) of (5.1.8) and (5.1.1), we obtain for $i = 2$

$$(t_j t_{j+1})^{c_2} = t_j^{t_{1,j+1}t_{2,j+2}} t_{j+1}^{t_{1,j+1}t_{2,j+2}} = t_{1,j}^{t_{2,j+2}} z t_{1,j+2}^{t_{2,j+2}} = z t_{1,j} t_1.$$

Similarly, if $i > 2$, then

$$\begin{aligned} (t_j \cdots t_{j+i-1})^{c_i} &= (t_j \cdots t_{j+i-2})^{c_{i-1}t_{i,j+i}} t_{j+i-1}^{c_i} \\ &= z^{i-2} (t_{1,j} t_1 t_2 \cdots t_{i-2})^{t_{i,j+i}} t_{j+i-1}^{c_{i-2}t_{i-1,j+i-1}t_{i,j+i}} \\ &= z^{i-1} t_{1,j} t_1 t_2 \cdots t_{i-2} t_{i-1}. \end{aligned}$$

As $t_{i,j} t_i = z t_i t_{i+1,j}$ for $j \geqslant i+2$ by part (5) of (5.1.8), we have $t_{1,j} t_1 = z t_1 t_{2,j}$. Thus

$$t_{1,j} t_1 \cdots t_{i-2} t_{i-1} = z^{i-2} t_1 \cdots t_{i-2} \cdot t_{i-1,j} t_{i-1} = z^{i-1} t_1 \cdots t_{i-1} t_{i,j}. \qquad \blacksquare$$

(5.1.10) Lemma. Set $c_i := t_{1,j+1} t_{2,j+2} \cdots t_{i,j+i}$ for $i, j \in \{1, \ldots, n\}$ with $i < j$. Then $t_{(i,j)}^{c_i} = z^{(i-1)(j-1)} t_{(j,i)}$ for $(i,j) \in \mathcal{C}(i+j)$.

Proof. Suppose first that $i = 1$. As $t_{(1)} = 1$, we use (5.1.1) and part (4) of (5.1.8) to get

$$t_{(1,j)}^{c_1} = (t_2 t_3 \cdots t_j)^{t_{1,j+1}} = (t_2 t_3 \cdots t_{j-1})^{t_{1,j+1}} t_j^{t_{1,j+1}} = z^{j-2} t_2 \cdots t_{j-1} t_{1,j}.$$

By part (1) of (5.1.8), $t_{1,j} = t_{j,1} = t_{j-1} t_{j-2} \cdots t_1 \cdots t_{j-1}$, and therefore

$$t_{(1,j)}^{c_1} = z^{2(j-2)} t_1 \cdots t_{j-1} = t_{(j,1)}.$$

Now let $i \geqslant 2$. Note that c^π and $(t_{i+1} t_{i+2} \cdots t_{j-1})^\pi$ operate on the disjoint sets $\{1, 2, \ldots, i, j+1, j+2, \ldots, j+i\}$ and $\{i+1, i+2, \ldots j-1, j\}$, respectively. Using (5.1.1) and

(5.1.9), we get

$$\begin{aligned}
t_{(i,j)}^{c_i} &= (t_1 \cdots t_{i-1})^{c_i}(t_{i+1} \cdots t_{i+j-1})^{c_i} \\
&= (t_{j+1} \cdots t_{j+i-1})(t_{i+1} \cdots t_{j-1})^{c_i}(t_j \cdots t_{i+j-1})^{c_i} \\
&= (t_{j+1} \cdots t_{j+i-1})z^{i(j-i-1)}(t_{i+1} \cdots t_{j-1})(t_j \cdots t_{i+j-1})^{c_i} \\
&= (t_{j+1} \cdots t_{j+i-1})z^{i(j-i-1)}(t_{i+1} \cdots t_{j-1})(t_1 \cdots t_{i-1})t_{i,j} \\
&= z^{(j-i-1)}(t_{j+1} \cdots t_{j+i-1})(t_1 \cdots t_{i-1})(t_{i+1} \cdots t_{j-1})t_{j,i} \\
&= (t_{j+1} \cdots t_{j+i-1})(t_1 \cdots t_{i-1}t_i t_{i+1} \cdots t_{j-1}) \\
&= z^{(i-1)(j-1)}(t_1 \cdots t_{j-1})(t_{j+1} \cdots t_{j+i-1}) = z^{(i-1)(j-1)}t_{(j,i)}. \qquad \blacksquare
\end{aligned}$$

Recall $t(i,\ell) := t_{\ell+1}t_{\ell+2} \cdots t_{\ell+i-1}$ from (5.1.2). Then, by definition, $t_{(i,j)} = t(i,0)t(j,i)$ and $t_{(j,i)} = t(j,0)t(i,j)$. The following generalization of (5.1.10) is straightforward:

(5.1.11) Lemma. Let $i, j, k \in \{1, \ldots, n\}$ be such that $i < j$ and set

$$c_i := t_{k,j+k}t_{k+1,j+k+1} \cdots t_{k+i-1,j+k+i-1}.$$

Then $(t(i,k-1)t(j,i+k-1))^{c_i} = z^{(i-1)(j-1)}t(j,k-1)t(i,j+k-1)$. $\blacksquare$

Finally, we can complete this subsection by giving the

Proof of (5.1.6). If $\Lambda \notin \mathcal{O}(n) \cup \mathcal{D}^-(n)$, then the claim is trivially true as $t_\Lambda \sim_{\tilde{S}_n} zt_\Lambda$. If $\lambda = \Lambda$, then there is nothing to prove. Consequently, suppose that $\lambda \neq \Lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$. It is sufficient to prove the claim for one "shuffle" $(\lambda_l, \lambda_{l+1}) \subseteq \lambda \to (\lambda_{l+1}, \lambda_l) \subseteq \Lambda$. Additionally, we may assume that this is the only shuffle which has to be done, as interchanging λ_l with λ_{l+1} yields again a composition of n. Let $(\lambda_l, \lambda_{l+1}) = (i,j)$ be such that $i < j$. Recall (5.1.2). Set $k := \ell_l + 1$ and $c := t_{k,j+k}t_{k+1,j+k+1} \cdots t_{k+i-1,j+k+i-1}$. Note that c^π acts on $\{\ell_l + 1, \ldots, \ell_{l+1}\} \cup \{\lambda_{l+1} + \ell_l + 1, \ldots, \ell_{l+2}\}$, whereas $t(\lambda_r, \ell_r)^\pi$ acts on the disjoint set $\{\ell_r + 1, \ldots, \ell_{r+1}\}$ provided $r \notin \{l, l+1\}$. Thus, by (5.1.1) and (5.1.11),

$$t_\lambda^c = \left(\prod_{r=1}^{l-1} t(\lambda_r, \ell_r)^c\right)(t(i,k-1)t(j,i+k-1))^c \left(\prod_{r=l+2}^{l(\lambda)} t(\lambda_r, \ell_r)^c\right) = z^{s+(i-1)(j-1)}t_\Lambda$$

where

$$s = \sum_{r \neq l, l+1} i(\lambda_r - 1) = i(n - i - j - l(\lambda) + 2) \equiv i(i + j + n + l(\lambda)) \bmod 2.$$

Suppose that $\Lambda \in \mathcal{O}(n)$. Then $i \equiv j \equiv 1 \bmod 2$ and the signature $n - l(\lambda)$ is even. Thus $s + (i-1)(j-1) \equiv 0 \bmod 2$. On the other hand, if $\Lambda \in \mathcal{D}^-(n)$, then $s \equiv ij \bmod 2$. In this case, clearly $t_\lambda^c = zt_\Lambda$ if i and j have the same parity, and $t_\lambda^c = t_\Lambda$ otherwise. ∎

Recall (5.1.3) where $\lambda = (4, 1, 3, 7)$. We shuffle

$$\lambda \xrightarrow{z} (4, 1, 7, 3) \xrightarrow{z} (4, 7, 1, 3) \to (7, 4, 1, 3) \xrightarrow{z} (7, 4, 3, 1) = \Lambda.$$

Thus t_λ and zt_Λ are conjugate in $\tilde{S}_{15}$.
In (5.4.8), we will work out an analogue to (5.1.6) in $\tilde{A}_n$.

A technique for the trace formula

(5.1.12) Definition. Let $x = (i_1, \dots, i_m) \in S_n$ such that $i_1 = \min\{i_1, \dots, i_m\}$. We define the lift $\tilde{x} := t_{i_1,i_m} t_{i_2,i_m} \cdots t_{i_{m-1},i_m} \in \tilde{S}_n$ of x. Moreover, if $x = x_1 \cdots x_k$ is a product of pairwise disjoint cycles $x_1, \dots, x_k$ such that $x_j = (i_{j_1}, \dots, i_{j_m})$ with $i_{j_1} = \min\{i_{j_1}, \dots, i_{j_m}\}$ for all $j \in \{1, \dots, k\}$ and $i_{1_1} < i_{2_1} < \cdots < i_{k_1}$, we set $\tilde{x} := \tilde{x}_1 \cdots \tilde{x}_k$.

Recall (3.13). We can determine the distribution of the elements of a coset $\tilde{x}K$ for $x \in S_n$ over the conjugacy classes of $\tilde{S}_n$ or $\tilde{A}_n$ as follows. Here K is a subgroup of $\tilde{S}_n$ such that $z \notin K$ (otherwise, $Ve_K = \mathrm{Fix}_V(K) = 0$ for every spin module V). Hence $K \cong K^\pi \leqslant S_n$. We suppose that π is explicitly known here. For example, we start with $\langle k_1, \dots, k_m \rangle \leqslant S_n$ and take $K := \langle K_1, \dots, K_m \rangle$ where $K_i \in \tilde{S}_n$ is a suitable lift of k_i. Then $\pi|_K$ is completely determined by $K_i^\pi = k_i$ for all $i \in \{1, \dots, m\}$. Let $x \in S_n$ and $k \in K$. First, we determine the cycle type λ of xk^π. If λ does not characterize a split class of $\tilde{S}_n$, that is, $t_\lambda \sim_{\tilde{S}_n} zt_\lambda$, then $\tilde{x}k \sim_{\tilde{S}_n} t_\lambda$. Otherwise, it is easy to find an element $g \in S_n$ that conjugates xk^π to $s_\lambda := t_\lambda^\pi$. As

$$((\tilde{x}k)^{\tilde{g}})^\pi = (\tilde{x}^\pi k^\pi)^{\tilde{g}^\pi} = (xk^\pi)^g = s_\lambda,$$

we have $(\tilde{x}k)^{\tilde{g}} \in \{t_\lambda, zt_\lambda\}$. Having a faithful representation δ of $\tilde{S}_n$ at hand (for instance, a basic spin representation, see Section 6), it is now straightforward to check whether $((\tilde{x}k)^{\tilde{g}})^\delta = t_\lambda^\delta$ or $((\tilde{x}k)^{\tilde{g}})^\delta = (zt_\lambda)^\delta$.

5.2 Young subgroups

For each $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{C}(n)$ there is a Young subgroup of S_n,

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_l}$$

where $S_{\lambda_i} = S_{\{\ell_i+1,\ldots,\ell_{i+1}\}} \leqslant S_n$ with $\ell_i = \lambda_1 + \cdots + \lambda_{i-1}$ for all $i \in \{1, \ldots, l\}$. Similarly, $A_\lambda := S_\lambda \cap A_n$ is a Young subgroup of A_n. Thus we have subgroups $\tilde{S}_\lambda := S_\lambda^{\pi^{-1}}$ and $\tilde{A}_\lambda := A_\lambda^{\pi^{-1}} = \tilde{S}_\lambda \cap \tilde{A}_n$ which we refer to as *Young subgroups* of $\tilde{S}_n$ or $\tilde{A}_n$, respectively. We also call Y_λ a Young subgroup of $\tilde{S}_n$ whenever $\tilde{A}_\lambda \leqslant Y_\lambda \leqslant \tilde{S}_\lambda$. As conjugacy of $\tilde{S}_n$-subgroups containing z is inherited from the conjugacy of their π-images in S_n, we see that each Young subgroup Y_λ is conjugate in $\tilde{S}_n$ to a *standard* Young subgroup Y_Λ where $\Lambda = \lambda_\geqslant \in \mathcal{P}(n)$. It follows that $\tilde{A}_\lambda$ and $\tilde{A}_\Lambda$ are conjugate in $\tilde{A}_n$. Hence we may equally assume that $\lambda \in \mathcal{P}(n)$ and neglect the term “standard”.

Conjugacy classes

Since Y_λ^π is a direct product, its conjugacy classes can be parameterized by tuples of parameters of the component groups. For instance, a conjugacy class of S_λ corresponds to an l-tuple $(\lambda^1, \ldots, \lambda^l)$ such that $\lambda^i \in \mathcal{P}(\lambda_i)$ for all $i \in \{1, \ldots, l\}$. For any two partitions μ and ν, denote by $\mu \sqcup \nu$ the concatenation of μ and ν. The following theorem characterizes the S_λ-classes which split in $\tilde{S}_\lambda$:

(5.2.1) Theorem ([68, (2.6)]). For $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{C}(n)$, let C be a conjugacy class of type $\lambda^* := \lambda^1 \sqcup \ldots \sqcup \lambda^l$ of S_λ, and set $\Lambda^* := \lambda^*_\geqslant \in \mathcal{P}(n)$. Then $C^{\pi^{-1}}$ splits as the union of two distinct conjugacy classes of $\tilde{S}_\lambda$ if and only if either $\Lambda^* \in \mathcal{O}(n)$, or Λ^* is odd and $\lambda^i \in \mathcal{D}(\lambda_i)$ for all $i \in \{1, \ldots, l\}$.

We are going to discuss the construction of (ordinary and p-modular) character tables of Young subgroups Y_λ of $\tilde{S}_n$, as we aim to generate characters for $\tilde{S}_n$ or $\tilde{A}_n$ by inducing characters of Y_λ. By transitivity of induction, we may restrict ourselves to the case where $\lambda = (k, l)$ with $k + l = n$.
In the remainder of this subsection, fix k and l with $2 \leqslant k, l \leqslant n-2$ such that $(k, l) \in \mathcal{P}(n)$. We then simply set $\tilde{S}_{k,l} := \tilde{S}_{(k,l)}$ and $\tilde{A}_{k,l} := \tilde{A}_{(k,l)}$. Note that

$$\tilde{S}_{k,l} = \langle t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n-1} \rangle.$$

We also refer to $\tilde{S}_k = \langle t_1, \ldots, t_{k-1} \rangle$ and, abusing the notation, $\tilde{S}_l := \langle t_{k+1}, \ldots, t_{n-1} \rangle$. This shall imply $S_m = \tilde{S}_m^\pi$ and $A_m = \tilde{A}_m^\pi$ with $\tilde{A}_m = \tilde{S}_m \cap \tilde{A}_n$ for $m \in \{k, l\}$.
We will work out an explicit description of the character tables of $\tilde{S}_{k,l}$ and $\tilde{A}_{k,l}$, provided that we know the corresponding tables of $\tilde{S}_k$, $\tilde{S}_l$, and $\tilde{A}_k$, $\tilde{A}_l$. Besides $\tilde{S}_{k,l}$ and $\tilde{A}_{k,l}$, we

shall be interested in the Young subgroups

$$\tilde{A}_k \circ \tilde{S}_l := (A_k \times S_l)^{\pi^{-1}} \quad \text{and} \quad \tilde{S}_k \circ \tilde{A}_l := (S_k \times A_l)^{\pi^{-1}}.$$

Assume that Y is one of the groups $\tilde{A}_{k,l}$, $\tilde{A}_k \circ \tilde{S}_l$, or $\tilde{S}_k \circ \tilde{A}_l$, and let X be $\tilde{A}_k \times \tilde{A}_l$, $\tilde{A}_k \times \tilde{S}_l$, or $\tilde{S}_k \times \tilde{A}_l$, respectively. Denote by Z the central subgroup $\langle (z, z) \rangle$ of X. As (5.1.1) implies

$$[\tilde{A}_k, \tilde{A}_l] = [\tilde{A}_k, \tilde{S}_l] = [\tilde{S}_k, \tilde{A}_l] = 1,$$

we see that $Y \cong X/Z$ (justifying our notation of central products using $\circ$). From (5.2.1) we get at once:

(5.2.2) Corollary. Let D be a conjugacy class of type (λ, μ) of $S_{k,l}$. Then $\tilde{D} := D^{\pi^{-1}}$ splits in $\tilde{S}_{k,l}$ if and only if (λ, μ) is contained in $\mathcal{O}(k) \times \mathcal{O}(l)$, $\mathcal{D}^+(k) \times \mathcal{D}^-(l)$, or $\mathcal{D}^-(k) \times \mathcal{D}^+(l)$. In particular, for $x \in \tilde{D}$ we have $|C_{\tilde{S}_{k,l}}(x)| = 2|C_{S_{k,l}}(x^\pi)|$ if $\tilde{D}$ splits, and we have $|C_{\tilde{S}_{k,l}}(x)| = |C_{S_{k,l}}(x^\pi)|$ otherwise. ∎

Write $Y = Y_1 \circ Y_2$ for any of the central products $\tilde{A}_{k,l}$, $\tilde{A}_k \circ \tilde{S}_l$, or $\tilde{S}_k \circ \tilde{A}_l$, and let $x_i, y_i \in Y_i$ for $i \in \{1, 2\}$. Then $x_1x_2 = y_1y_2$ if and only if $x_i = y_i$ or $x_i = zy_i$ for $i \in \{1, 2\}$. Moreover, $(x_1x_2)^{y_1y_2} = x_1^{y_1}x_2^{y_2}$. Hence x_1x_2 and zx_1x_2 are conjugate in Y if and only if x_1 and zx_1 are conjugate in Y_1, or x_2 and zx_2 are conjugate in Y_2 (or both). Equivalently, x_1x_2 and zx_1x_2 are not conjugate in Y if and only if neither x_1 and zx_1 are conjugate in Y_1 nor x_2 and zx_2 are conjugate in Y_2. Thus (5.1.4) yields the following.

(5.2.3) Corollary. (Recall: for $m \in \{k, l\}$ an A_m-class of type $\tau \in \mathcal{O}(m) \cap \mathcal{D}^+(m)$ means one of the two A_m-classes of type $\tau^\pm$.)

(1) Let C be a conjugacy class of $A_k \times A_l$ of type (λ, μ). Then $C^{\pi^{-1}}$ splits in $\tilde{A}_{k,l}$ if and only if $(\lambda, \mu) \in (\mathcal{O}(k) \cup \mathcal{D}^+(k)) \times (\mathcal{O}(l) \cup \mathcal{D}^+(l))$.

(2) Let C be a conjugacy class of $A_k \times S_l$ of type (λ, μ). Then $C^{\pi^{-1}}$ splits in $\tilde{A}_k \circ \tilde{S}_l$ if and only if $(\lambda, \mu) \in (\mathcal{O}(k) \cup \mathcal{D}^+(k)) \times (\mathcal{O}(l) \cup \mathcal{D}^-(l))$.

(3) Let C be a conjugacy class of $S_k \times A_l$ of type (λ, μ). Then $C^{\pi^{-1}}$ splits in $\tilde{S}_k \circ \tilde{A}_l$ if and only if $(\lambda, \mu) \in (\mathcal{O}(k) \cup \mathcal{D}^-(k)) \times (\mathcal{O}(l) \cup \mathcal{D}^+(l))$. ∎

Element orders and powermaps

Let $m \geqslant 2$ and $x = \prod_{i \in I} t_i$, $y = \prod_{j \in J} t_j$ such that x^π and y^π act on disjoint subsets of

$\{1, \ldots, n\}$. Using (5.1.1), induction over m proves

$$(xy)^m = z^{|I| \cdot |J| \cdot m(m-1)/2} x^m y^m.$$

Let m be the least common multiple of the orders $o(x^\pi)$ and $o(y^\pi)$, so that $o((xy)^\pi) = m$. Then $o := o(xy) \in \{m, 2m\}$. More precisely,

(1) if x and y are odd ($|I| \equiv |J| \equiv 1 \bmod 2$),

  (a) if $m(m-1)/2$ is even ($m \equiv 0$ or $1 \bmod 4$), then

    (i) if $x^m = y^m = 1$ or $x^m = y^m = z$, then $o = m$;

    (ii) otherwise, $o = 2m$.

  (b) if $m(m-1)/2$ is odd ($m \equiv 2$ or $3 \bmod 4$), then

    (i) if $x^m = 1$ and $y^m = z$, or $x^m = z$ and $y^m = 1$, then $o = m$;

    (ii) otherwise, $o = 2m$.

(2) otherwise, if x or y is even,

  (a) if $x^m = y^m = 1$ or $x^m = y^m = z$, then $o = m$;

  (b) otherwise, $o = 2m$.

The above equation also determines the q-th powermap $x \mapsto x^q$ for a rational prime q: we have $(xy)^q = x^q y^q$ unless both x and y are odd. In the latter case,

$$(xy)^q = z^{q(q-1)/2} x^q y^q = \begin{cases} z x^q y^q & \text{if } q-1 \equiv 2 \bmod 4, \text{ or } q = 2, \\ x^q y^q & \text{if } q-1 \equiv 0 \bmod 4. \end{cases}$$

Characters

Let $F \in \{\mathcal{F}, f\}$ be as in Section 1 for $G = \tilde{S}_n$. We keep $Y = Y_1 \circ Y_2$ and write correspondingly $X = X_1 \times X_2$. Denote by $\mathcal{I}(S)$ the set of characters afforded by the irreducible F-representations of $S \leqslant \tilde{S}_n$, that is, $\mathcal{I}(S) = \mathrm{Irr}(S)$ if $F = \mathcal{F}$ or $\mathcal{I}(S) = \mathrm{IBr}(S)$ if $F = f$. Let $\mathcal{I}(S)^\pm := \{\phi \in \mathcal{I}(S) : \phi(z) = \pm\phi(1)\}$. Then the characters afforded by the irreducible F-representations of X are exactly the outer tensor products $\varphi \otimes \psi$ for $\varphi \in \mathcal{I}(X_1)$ and $\psi \in \mathcal{I}(X_2)$. As $(\varphi \otimes \psi)(1,1) = (\varphi \otimes \psi)(z,z)$ if $\varphi \in \mathcal{I}(X_1)^\epsilon$ and $\psi \in \mathcal{I}(X_2)^\epsilon$ for $\epsilon \in \{+,-\}$, the character $\varphi \otimes \psi$ can be pushed forward to the factor group $X/Z \cong Y$, yielding an irreducible character $\varphi \circledcirc \psi$ of Y. Thus we may identify

$$\mathcal{I}(Y) = \{\, \varphi \circledcirc \psi : \varphi \in \mathcal{I}(X_1)^+, \psi \in \mathcal{I}(X_2)^+ \,\} \cup \{\, \varphi \circledcirc \psi : \varphi \in \mathcal{I}(X_1)^-, \psi \in \mathcal{I}(X_2)^- \,\}$$

and the character table of Y can be constructed straightforwardly. Now this enables us to construct all spin characters of $\tilde{S}_{k,l}$ by means of Clifford's theory, see (1.4), (1.5) and (1.6). Let χ be an irreducible character of $\tilde{S}_{k,l}$ with $\chi(z) = -\chi(1)$ and assume that $\varphi \circledcirc \psi \in \mathcal{I}(\tilde{A}_{k,l})$ is a constituent of $\chi\downarrow_{\tilde{A}_{k,l}}$. We are interested in the inertia subgroup $T := I_{\tilde{S}_{k,l}}(\varphi \circledcirc \psi)$. We have $\tilde{A}_{k,l} \trianglelefteq T \trianglelefteq \tilde{S}_{k,l}$ and $\tilde{S}_{k,l}/\tilde{A}_{k,l} \cong \langle t_1^\pi, t_{k+1}^\pi \rangle \cong 2^2$. Hence

$$T \in \{\, \tilde{A}_{k,l},\ \tilde{S}_k \circ \tilde{A}_l,\ \langle \tilde{A}_{k,l}, t_1 t_{k+1} \rangle,\ \tilde{A}_k \circ \tilde{S}_l,\ \tilde{S}_{k,l} \,\}.$$

Moreover, for $t \in \tilde{S}_{k,l}$ we have $(\varphi \circledcirc \psi)^t = \varphi \circledcirc \psi$ if and only if $(\varphi \otimes \psi)^t = \varphi \otimes \psi$, that is, $\varphi^t = \varphi$ and $\psi^t = \psi$. Recall that $\varphi^{t_{k+1}} = \varphi$ for all $\varphi \in \mathcal{I}(\tilde{A}_k)$, and $\psi^{t_1} = \psi$ for all $\psi \in \mathcal{I}(\tilde{A}_l)$. Then $t_1 t_{k+1} \in T$ if and only if $(\varphi \circledcirc \psi)^{t_1 t_{k+1}} = \varphi^{t_1} \circledcirc \psi^{t_{k+1}} = \varphi \circledcirc \psi$, that is, both t_1 and t_{k+1} are contained in T. In particular, $T \neq \langle \tilde{A}_{k,l}, t_1 t_{k+1} \rangle$ and we are in one of the following cases:

(1) $T = \tilde{A}_{k,l}$ and

$$\begin{aligned} \chi\downarrow_{\tilde{A}_{k,l}} &= \varphi \circledcirc \psi + (\varphi \circledcirc \psi)^{t_1} + (\varphi \circledcirc \psi)^{t_{k+1}} + (\varphi \circledcirc \psi)^{t_1 t_{k+1}} \\ &= \varphi \circledcirc \psi + \varphi^{t_1} \circledcirc \psi + \varphi \circledcirc \psi^{t_{k+1}} + \varphi^{t_1} \circledcirc \psi^{t_{k+1}}. \end{aligned}$$

Then $(\varphi \circledcirc \psi)\uparrow^{\tilde{S}_{k,l}} = \chi$.

(2) $T = \tilde{S}_k \circ \tilde{A}_l$ and $\chi\downarrow_{\tilde{A}_{k,l}} = \varphi \circledcirc \psi + \varphi \circledcirc \psi^{t_{k+1}}$. As this implies $I_{\tilde{S}_k \circ \tilde{A}_l}(\varphi \circledcirc \psi) = \tilde{S}_k \circ \tilde{A}_l$, there exists some $\xi \in \mathcal{I}(\tilde{S}_k \circ \tilde{A}_l)$ with $\xi\downarrow_{\tilde{A}_{k,l}} = \varphi \circledcirc \psi$. Indeed, $\xi = \hat{\varphi} \circledcirc \psi$ for some $\hat{\varphi} \in \{\phi, \phi^a\}$ that extends φ to $\tilde{S}_k$ where $\varphi\uparrow^{\tilde{S}_k} = \phi + \phi^a$. We see that $\xi^{t_{k+1}} \neq \xi$ and therefore $\xi\uparrow^{\tilde{S}_{k,l}} = \chi$.

(3) Analogously to (2), we have $T = \tilde{A}_k \circ \tilde{S}_l$ and $\chi\downarrow_{\tilde{A}_{k,l}} = \varphi \circledcirc \psi + \varphi^{t_1} \circledcirc \psi$. Now there exists some $\xi \in \mathcal{I}(\tilde{A}_k \circ \tilde{S}_l)$ with $\xi\downarrow_{\tilde{A}_{k,l}} = \varphi \circledcirc \psi$ and $\xi\uparrow^{\tilde{S}_{k,l}} = \chi$.

(4) $T = \tilde{S}_{k,l}$ with two possibilites: either $\chi\downarrow_{\tilde{A}_{k,l}} = \varphi \circledcirc \psi$, or $\chi\downarrow_{\tilde{A}_{k,l}} = 2(\varphi \circledcirc \psi)$, but we are going to show that only the latter can actually arise. Suppose that χ restricts irreducibly to $\tilde{A}_{k,l}$. Then its restrictions to $\tilde{S}_k \circ \tilde{A}_l$ and $\tilde{A}_k \circ \tilde{S}_l$ are irreducible as well. More precisely, $\chi\downarrow_{\tilde{S}_k \circ \tilde{A}_l} = \hat{\varphi} \circledcirc \psi$ and $\chi\downarrow_{\tilde{A}_k \circ \tilde{S}_l} = \varphi \circledcirc \hat{\psi}$ where

$\hat{\varphi} \in \mathcal{I}(\tilde{S}_k)$ and $\hat{\psi} \in \mathcal{I}(\tilde{S}_l)$ are extensions of φ and ψ, respectively. Let δ be a representation of $\tilde{S}_{k,l}$ over F that affords χ. We may write $\delta\downarrow_{\tilde{S}_k \circ \tilde{A}_l} = \delta_{\hat{\varphi}} \otimes \delta_\psi$ for suitable representations $\delta_{\hat{\varphi}}$ of $\tilde{S}_k$ and δ_ψ of $\tilde{A}_l$ affording $\hat{\varphi}$ and ψ, respectively, and analogously $\delta\downarrow_{\tilde{A}_k \circ \tilde{S}_l} = \delta_\varphi \otimes \delta_{\hat{\psi}}$. It follows that

$$(t_1 t_{k+1})^\delta = (t_1^{\delta_{\hat{\varphi}}} \otimes 1^{\delta_\psi})(1^{\delta_\varphi} \otimes t_{k+1}^{\delta_{\hat{\psi}}}) = t_1^{\delta_{\hat{\varphi}}} \otimes t_{k+1}^{\delta_{\hat{\psi}}} = (1^{\delta_\varphi} \otimes t_{k+1}^{\delta_{\hat{\psi}}})(t_1^{\delta_{\hat{\varphi}}} \otimes 1^{\delta_\psi}) = (t_{k+1} t_1)^\delta,$$

in contradiction to $t_1 t_{k+1} = z t_{k+1} t_1$ and $z^\delta = -1$. Thus we have $\chi\downarrow_{\tilde{A}_{k,l}} = 2(\varphi \circledcirc \psi)$. Furthermore, by (1.5), χ vanishes on $\tilde{S}_{k,l} \setminus \tilde{A}_{k,l}$.

Hence we have to determine the inertia subgroup $T = I_{\tilde{S}_{k,l}}(\varphi \circledcirc \psi)$ for $\varphi \circledcirc \psi \in \mathcal{I}(\tilde{A}_{k,l})$. Since $t_1 \in T$ if and only if $\varphi^{t_1} = \varphi$, and $t_{k+1} \in T$ if and only if $\psi^{t_{k+1}} = \psi$, this can be done easily.

5.3 Labels for characters II

Let $\mathcal{R}_p(n)$ be the set of p-regular partitions λ of n, that is, no part of λ is repeated more than $p-1$ times. Then $\mathrm{Irr}(S_n)$ and $\mathrm{IBr}(S_n)$ are in bijection with $\mathcal{P}(n)$ and $\mathcal{R}_p(n)$, respectively. The following result on the decomposition matrix of S_n is also well-known:

(5.3.1) Theorem (James [26, (12.3)]). Let $\mathcal{R}_p^{\trianglerighteq}(n) = (\lambda_1, \dots \lambda_r)$ be the list of p-regular partitions of n ordered with respect to the (reverse) lexicographical order $\trianglerighteq$ on $\mathcal{P}(n)$. Moreover, let $\mathcal{P}(n) \setminus \mathcal{R}_p^{\trianglerighteq}(n) = (\mu_1, \dots \mu_s)$. Then the p-modular decomposition matrix of S_n can be written in the form

	λ_1	λ_2	$\dots$	$\dots$	λ_r
λ_1	1				
λ_2	$\star$	1			
$\vdots$	$\star$	$\star$	1		
$\vdots$	$\vdots$	$\vdots$		$\ddots$	
λ_r	$\star$	$\star$	$\dots$	$\star$	1
μ_1	$\star$	$\star$	$\dots$	$\dots$	$\star$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
μ_s	$\star$	$\star$	$\dots$	$\dots$	$\star$

(5.3.2) Now it is easy to match the elements of $\mathrm{IBr}(S_n)$ with partitions from $\mathcal{R}_p^{\triangleright}(n)$. Assume that we know the labelling of $\mathrm{Irr}(S_n)$ in terms of $\mathcal{P}(n)$. Put the subset of elements of $\mathrm{Irr}(S_n)$ that corresponds to $\mathcal{R}_p(n)$ in reverse lexicographical order and compute the p-modular decomposition of these characters in terms of $\mathrm{IBr}(S_n)$. Sort the columns of the resulting decomposition matrix reverse-lexicographically, using a permutation α say, and apply α to $\mathrm{IBr}(S_n)$ (interpreted as a list). Then the ith element $\phi_i \in \mathrm{IBr}(S_n)^\alpha$ corresponds to the ith element $\lambda_i \in \mathcal{R}_p^{\triangleright}(n)$, and we write $\phi_i = [\lambda_i]$. Moreover, we agree on labelling $\phi \in \mathrm{IBr}(A_n)$ as follows: if $[\lambda]{\downarrow_{A_n}} = \phi = [\mu]{\downarrow_{A_n}}$ for $\lambda \triangleright \mu$, then ϕ is labelled by λ. Otherwise, if $[\lambda]{\downarrow_{A_n}} = \phi + \phi^c$ with $\phi > \phi^c$, then ϕ is labelled by $[\lambda, +]$ whereas ϕ^c is labelled by $[\lambda, -]$, see 4.3.

In general, we denote the characters of subgroups $X \leqslant \tilde{S}_n$ here as follows. Let

$$\|\nu\| := \begin{cases} \langle\nu\rangle & \text{if } \nu \text{ labels } \chi \in \mathrm{Irr}(X) \text{ with } \chi(z) = \chi(1), \\ [\nu] & \text{if } \nu \text{ labels } \phi \in \mathrm{IBr}(X) \text{ with } \phi(z) = \phi(1), \\ \langle\!\langle\nu\rangle\!\rangle & \text{if } \nu \text{ labels } \chi \in \mathrm{Irr}(X) \text{ with } \chi(z) = -\chi(1), \\ [\![\nu]\!] & \text{if } \nu \text{ labels } \phi \in \mathrm{IBr}(X) \text{ with } \phi(z) = -\phi(1). \end{cases}$$

If $\varphi = \|\lambda\|$ is a character of $\tilde{A}_k$ and $\psi = \|\mu\|$ is a character of $\tilde{A}_l$, then it is straightforward to label the character $\varphi \circledcirc \psi$ of $\tilde{A}_{k,l}$ by the pair (λ, μ). We write

$$\varphi \circledcirc \psi = \|\lambda : \mu\|.$$

Here it is to be understood that λ and μ may stand for "signed" parameters of pairs of associate characters, that is, $\lambda \in \{\lambda_\circ, (\lambda_\circ, \varepsilon)\}$ and $\mu \in \{\mu_\circ, (\mu_\circ, \tau)\}$ where $\lambda_\circ \in \mathcal{P}(k)$, $\mu_\circ \in \mathcal{P}(l)$, and $\varepsilon, \tau \in \{+, -\}$. Similarly, we may label the characters of $\tilde{S}_k \circ \tilde{A}_l$ and $\tilde{A}_k \circ \tilde{S}_l$, as well as the irreducible characters χ of $\tilde{S}_{k,l}$ with $\chi(z) = \chi(1)$. If χ is a character of $\tilde{S}_{k,l}$ with $\chi(z) = -\chi(1)$, and $\varphi \circledcirc \psi = \|\lambda : \mu\|$ is an irreducible constituent of $\chi{\downarrow_{\tilde{A}_{k,l}}}$, then we label χ as in Table 5.1. Compare the construction of χ in 5.2.

Table 5.1: Labels for irreducible characters of Young subgroups

	$I_{\tilde{S}_{k,l}}(\varphi \circledcirc \psi)$				Label for χ
(1)	$\tilde{A}_{k,l}$	$\chi\downarrow_{\tilde{A}_{k,l}}$	$=$	$\|\lambda_\circ,+:\mu_\circ,+\| + \|\lambda_\circ,-:\mu_\circ,+\|$ $+\|\lambda_\circ,+:\mu_\circ,-\| + \|\lambda_\circ,-:\mu_\circ,-\|$	$\|\lambda_\circ,\pm:\mu_\circ,\pm\|$
(2)	$\tilde{S}_k \circ \tilde{A}_l$	$\chi\downarrow_{\tilde{S}_k\circ\tilde{A}_l}$	$=$	$\|\lambda_\circ,\varepsilon:\mu_\circ,+\| + \|\lambda_\circ,-\varepsilon:\mu_\circ,-\|$	$\|\lambda_\circ,\varepsilon:\mu_\circ\|$
		$\chi\downarrow_{\tilde{A}_{k,l}}$	$=$	$\|\lambda_\circ:\mu_\circ,+\| + \|\lambda_\circ:\mu_\circ,-\|$	
		$\chi\downarrow_{\tilde{A}_k\circ\tilde{S}_l}$	$=$	$\|\lambda_\circ:\mu_\circ\|$	
(3)	$\tilde{A}_k \circ \tilde{S}_l$	$\chi\downarrow_{\tilde{A}_k\circ\tilde{S}_l}$	$=$	$\|\lambda_\circ,+:\mu_\circ,\tau\| + \|\lambda_\circ,-:\mu_\circ,-\tau\|$	$\|\lambda_\circ:\mu_\circ,\tau\|$
		$\chi\downarrow_{\tilde{A}_{k,l}}$	$=$	$\|\lambda_\circ,+:\mu_\circ\| + \|\lambda_\circ,-:\mu_\circ\|$	
		$\chi\downarrow_{\tilde{S}_k\circ\tilde{A}_l}$	$=$	$\|\lambda_\circ:\mu_\circ\|$	
(4)	$\tilde{S}_{k,l}$	$\chi\downarrow_{\tilde{A}_{k,l}}$	$=$	$2\|\lambda_\circ:\mu_\circ\|$	$\|\lambda_\circ:\mu_\circ\|$

5.4 Class Fusions

We determine various class fusions which will allow us to induce characters from subgroups of $\tilde{S}_n$ or $\tilde{A}_n$. Note that, if X and Y are conjugate subgroups of any group G, then $\mathcal{I}(X)\uparrow^G = \mathcal{I}(Y)\uparrow^G$ for $\mathcal{I} \in \{\mathrm{Irr}, \mathrm{IBr}\}$. Therefore it is sufficient to look at the following "default embeddings" $X \hookrightarrow G$ of the various groups here. Recall from (5.1.4) that $t_\lambda \nsim_{\tilde{S}_n} zt_\lambda$ if and only if $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$. Moreover, $t_\lambda \nsim_{\tilde{A}_n} zt_\lambda$ if and only if $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^+(n)$ including the cases $t_{\lambda^+} \nsim_{\tilde{A}_n} t_{\lambda^-}$ for $\lambda \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$. Note that $t_\lambda = t_{\lambda^+}$ by definition, and, without loss of generality, $t_{\lambda^-} = t_{\lambda^+}^{t_1}$.

(5.4.1) $\tilde{A}_n \hookrightarrow \tilde{S}_n$.
If $\lambda \in \mathcal{P}^+(n) \setminus (\mathcal{O}(n) \cup \mathcal{D}(n))$, then $\lambda \notin \mathcal{O}(n) \cap \mathcal{D}^+(n)$ and the classes of type λ in $\tilde{A}_n$ and $\tilde{S}_n$ coincide. Suppose that $\lambda \in (\mathcal{O}(n) \cup \mathcal{D}^+(n)) \setminus (\mathcal{O}(n) \cap \mathcal{D}^+(n))$. If $\lambda \in \mathcal{O}(n)$, then $t_\lambda \nsim_{\tilde{A}_n} zt_\lambda$ and $t_\lambda \nsim_{\tilde{S}_n} zt_\lambda$. If $\lambda \in \mathcal{D}(n)$, then $t_\lambda \nsim_{\tilde{A}_n} zt_\lambda$ but $t_\lambda \sim_{\tilde{S}_n} zt_\lambda$. Finally, if $\lambda \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$, then $t_{\lambda^\pm} \nsim_{\tilde{A}_n} zt_{\lambda^\pm}$ and $t_\lambda \nsim_{\tilde{S}_n} zt_\lambda$, but $t_\lambda = t_{\lambda^+} \sim_{\tilde{S}_n} t_{\lambda^-}$. Hence the first $\tilde{A}_n$-classes of types λ^+ and λ^- fuse in $\tilde{S}_n$ as the first class of type λ, and the second $\tilde{A}_n$-classes fuse correspondingly as the second class of type λ. ∎

If $\lambda \in \mathcal{P}(m)$ for $m < n$, we set $\Lambda(\lambda) := \lambda \sqcup (1^{n-m}) \in \mathcal{P}(n)$ and note that $t_{\Lambda(\lambda)} = t_\lambda$.

(5.4.2) $\tilde{S}_m \hookrightarrow \tilde{S}_n$ for $m < n$.

If $\lambda \in \mathcal{O}(m)$, then $\Lambda := \Lambda(\lambda) \in \mathcal{O}(n)$. Hence $t_\lambda \nsim_{\tilde{S}_m} zt_\lambda$ and $t_\Lambda = t_\lambda \nsim_{\tilde{S}_n} zt_\lambda = zt_\Lambda$. If $\lambda \in \mathcal{D}^-(m)$, then $t_\Lambda \sim_{\tilde{S}_n} zt_\Lambda$ unless $\Lambda \in \mathcal{D}^-(n)$, that is, $m = n-1$ and no part of λ equals 1. ∎

(5.4.3) $\tilde{A}_m \hookrightarrow \tilde{A}_n$ for $m < n$.

Assume that $\lambda \in (\mathcal{O}(m) \cup \mathcal{D}^+(m)) \setminus (\mathcal{O}(m) \cap \mathcal{D}^+(m))$. Then $t_\lambda \nsim_{\tilde{A}_n} zt_\lambda$ if and only if either $\lambda \in \mathcal{O}(m)$, or $\lambda \in \mathcal{D}^+(m)$, $m = n-1$, and no part of λ equals 1.

Suppose that $\lambda \in \mathcal{O}(m) \cap \mathcal{D}^+(m)$. Again, $\Lambda := \Lambda(\lambda) \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$ only if $m = n-1$ and 1 is not a part of λ, in this case there are four distinct classes of both $\tilde{A}_m$ and $\tilde{A}_n$ represented by $z^\varepsilon t_{\lambda^\pm} = z^\varepsilon t_{\Lambda^\pm}$ for $\varepsilon \in \{0,1\}$. Otherwise, $t_\Lambda = t_{\lambda^+} \sim_{\tilde{A}_n} t_{\lambda^-}$ and thus $zt_\Lambda = zt_{\lambda^+} \sim_{\tilde{A}_n} zt_{\lambda^-}$. ∎

(5.4.4) $\tilde{S}_{n-2} \hookrightarrow \tilde{A}_n$.

Let $X := \langle t_1 t_{n-1}, \dots, t_{n-3} t_{n-1} \rangle \leqslant \tilde{A}_n$. Then $X \cong \tilde{S}_{n-2}$ and the isomorphism reflects on the cycle types as follows. Let $s \in \tilde{S}_{n-2}$ be a standard representative of type λ. Then s corresponds to an element $t_\Lambda \in X$ of type

$$\Lambda := \begin{cases} \lambda \sqcup (1^2) & \text{if } \lambda \in \mathcal{P}^+(n-2), \\ \lambda \sqcup (2) & \text{if } \lambda \in \mathcal{P}^-(n-2). \end{cases}$$

We have $\Lambda_\geqslant \in \mathcal{O}(n)$ if and only if $\lambda \in \mathcal{O}(n-2)$, in which case clearly $t_{\Lambda_\geqslant} = t_\lambda \in \tilde{A}_n$. On the other hand, $\Lambda_\geqslant \in \mathcal{D}^+(n)$ if and only if $\lambda \in \mathcal{D}^-(n-2)$ and no part of λ equals 2. In this case, either $\lambda_{l(\lambda)} > 1$ and thus $\Lambda_\geqslant = \Lambda$, or $\lambda_{l(\lambda)} = 1$. If $\lambda = (\lambda_1, \dots, \lambda_l, 1)$, let $\lambda_* := (\lambda_1, \dots, \lambda_l)$. Note that $t_{\lambda_*}^\pi$ acts on $\{1, \dots, n-3\}$ and $t_\Lambda = t_{\lambda_*} t_{n-1}$. Then $t_{\Lambda_\geqslant} \sim_{\tilde{A}_n} t_\Lambda$ as $t_\Lambda^{t_{n-1}t_{n-2,n}} = t_{\lambda_*} t_{n-1}^{t_{n-1}t_{n-2,n}} = t_{\lambda_*} t_{n-2} = t_{\Lambda_\geqslant}$ (see (5.1.1) and (5.1.8)). In particular, note that the case $\Lambda_\geqslant \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$ can never occur. ∎

Young subgroups

Let $k, l \in \{2, \dots, n-2\}$ be such that $n = k + l$. Assume that $(\lambda, \mu) \in \mathcal{P}(k) \times \mathcal{P}(l)$ and define $\nu := \lambda \sqcup \mu \in \mathcal{C}(n)$. Let $\mu_* := (1^k) \sqcup \mu$, so that $t_\nu = t_\lambda t_{\mu_*}$ with $t_\lambda \in \langle t_1, \dots, t_{k-1} \rangle$ and $t_{\mu_*} \in \langle t_{k+1}, \dots, t_{n-1} \rangle$. Let C be a conjugacy class of $S_{k,l}$ having type (λ, μ) and set $\tilde{C} := C^{\pi^{-1}}$. If $\tilde{C}$ is a conjugacy class of $\tilde{S}_{k,l}$, then we define $t_\nu \in \tilde{C}$ to be its standard representative. If $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ splits, then we distinguish the first class $\tilde{C}_1$ from the second class $\tilde{C}_2$ by choosing the standard representatives $t_\nu \in \tilde{C}_1$ and $zt_\nu \in \tilde{C}_2$. Similar notions apply for other Young subgroups, where also signed parameters may be involved. In particular, we may assume that $t_{\lambda^-} = t_{\lambda^+}^{t_1}$ and $t_{\mu_*^-} = t_{\mu_*^+}^{t_{k+1}}$ where, by definition, $t_{\lambda^+} = t_\lambda$ and $t_{\mu_*^+} = t_{\mu_*}$. Recall (5.2.2) and (5.2.3).

(5.4.5) $\tilde{S}_{k,l} \hookrightarrow \tilde{S}_n$.
Apply (5.1.6) to determine whether $t_\nu \sim_{\tilde{S}_n} t_{\nu_\geqslant}$ or $t_\nu \sim_{\tilde{S}_n} zt_{\nu_\geqslant}$. ∎

(5.4.6) $\tilde{S}_k \circ \tilde{A}_l \hookrightarrow \tilde{S}_{k,l}$.
The split classes of $\tilde{S}_{k,l}$ are characterized by $\mathcal{O}(k)\times\mathcal{O}(l)$, $\mathcal{D}^-(k)\times\mathcal{D}^+(l)$, and $\mathcal{D}^+(k)\times\mathcal{D}^-(l)$ (note that $\mathcal{D}^+(k)\times\mathcal{D}^-(l)$ is not important to us since μ must be even here). Assume first that $(\lambda,\mu) \in \mathcal{O}(k)\times\mathcal{O}(l)$. If $\mu \notin \mathcal{D}^+(l)$, then there are two classes in both $\tilde{S}_k \circ \tilde{A}_l$ and $\tilde{S}_{k,l}$ represented correspondingly by $t_\lambda t_{\mu_*}$ and $zt_\lambda t_{\mu_*}$. If $(\lambda,\mu) \in \mathcal{D}^-(k)\times\mathcal{D}^+(l)$, then the same holds unless $\mu \in \mathcal{O}(l)$. Suppose now that $(\lambda,\mu) \in \mathcal{O}(k)\times(\mathcal{O}(l)\cap\mathcal{D}^+(l))$ or $(\lambda,\mu) \in \mathcal{D}^-(k)\times(\mathcal{O}(l)\cap\mathcal{D}^+(l))$. We have to deal with four distinct $\tilde{S}_k \circ \tilde{A}_l$-classes represented by $t_\lambda t_{\mu_*^+}$, $zt_\lambda t_{\mu_*^+}$, $t_\lambda t_{\mu_*^-}$, and $zt_\lambda t_{\mu_*^-}$. By definition, $t_{\mu_*^+} = t_{\mu_*}$, thus two corresponding classes of $\tilde{S}_{k,l}$ are clear. It remains to determine if $t_\lambda t_{\mu_*^-}$ is $\tilde{S}_{k,l}$-conjugate to $t_\lambda t_\mu$ or $zt_\lambda t_\mu$. By (5.1.1), we have $(t_\lambda t_{\mu_*^-})^{t_{k+1}} = t_\lambda^{t_{k+1}} t_\mu = t_\lambda t_\mu$ if λ is even, and $(t_\lambda t_{\mu_*^-})^{t_{k+1}} = zt_\lambda t_\mu$ if λ is odd. ∎

(5.4.7) $\tilde{A}_{k,l} \hookrightarrow \tilde{S}_k \circ \tilde{A}_l$.
The only critical classes are those represented by $t_{\lambda^-} t_\mu$. As t_μ is even and therefore centralized by t_1, we have $(t_{\lambda^-} t_\mu)^{t_1} = t_{\lambda^+} t_\mu = t_\lambda t_\mu$. ∎

Note that $\tilde{A}_{k,l} \hookrightarrow \tilde{S}_k \circ \tilde{A}_l \hookrightarrow \tilde{S}_{k,l}$ determines the fusion of $\tilde{A}_{k,l}$ in $\tilde{S}_{k,l}$. The cases $\tilde{A}_k \circ \tilde{S}_l \hookrightarrow \tilde{S}_{k,l}$ and $\tilde{A}_{k,l} \hookrightarrow \tilde{A}_k \circ \tilde{S}_l$ are handled analogously to $\tilde{S}_k \circ \tilde{A}_l \hookrightarrow \tilde{S}_{k,l}$ and $\tilde{A}_{k,l} \hookrightarrow \tilde{S}_k \circ \tilde{A}_l$, respectively.

(5.4.8) $\tilde{A}_{k,l} \hookrightarrow \tilde{A}_n$.
We work out an analogue to (5.1.6) in $\tilde{A}_n$. In order to avoid repetitions, identify $\lambda = \lambda^+$ and $\mu = \mu^+$ (as the corresponding standard representatives are equal). Then $t_{\lambda^+} t_{\mu_*^+} = t_\nu$ and $t_{\lambda^-} t_{\mu_*^-} = t_{\lambda^+}^{t_1} t_{\mu_*^+}^{t_{k+1}} = t_\nu^{t_1 t_{k+1}} \sim_{\tilde{A}_n} t_\nu$. Moreover, $t_{\lambda^+} t_{\mu_*^-} = t_\nu^{t_{k+1}}$ and $t_{\lambda^-} t_{\mu_*^+} = t_\nu^{t_1}$. We need to find the $\tilde{A}_n$-conjugacy class of t_ν (or possibly $t_\nu^{t_1}$, $t_\nu^{t_{k+1}}$) which is represented by $t_{\nu_\geqslant^\varepsilon}$ or $zt_{\nu_\geqslant^\varepsilon}$ where ε may be a sign. As in the proof of (5.1.6), a shuffle $(\lambda_i,\mu_j) \to (\mu_j,\lambda_i)$ can be achieved by conjugation with an element $c \in \tilde{S}_n$ having the same parity as λ_i, so that c is an element of $\tilde{A}_n$ if and only if λ_i is even. As the signature $n - l(\nu)$ is even, the conjugation by c produces a factor z^m where

$$m \equiv \lambda_i(\lambda_i + \mu_j + n + l(\nu)) + (\lambda_i - 1)(\mu_j - 1) \equiv \mu_j + 1 \bmod 2,$$

that is, we obtain a factor z everytime we shuffle an even μ_j in $\tilde{S}_n$. Let

$$e := \left|\{\,(\lambda_i, \mu_j) \subseteq \nu : \lambda_i < \mu_j \text{ and } \mu_j \equiv 0 \bmod 2\}\right|,$$
$$o := \left|\{\,(\lambda_i, \mu_j) \subseteq \nu : \lambda_i < \mu_j \text{ and } \lambda_i \equiv 1 \bmod 2\}\right|.$$

Here o counts the number of *odd* conjugating elements $c_1, \ldots, c_o$ used for shuffling from ν to $\nu_{\geqslant}$. In particular, $c_1 \cdots c_o \in \tilde{A}_n$ if and only if $o \equiv 0 \bmod 2$, in which case $t_\nu \sim_{\tilde{A}_n} z^e t_{\nu_\geqslant}$. If o is odd, then $t_\nu \sim_{\tilde{A}_n} z^e t^t_{\nu_\geqslant}$ for some $t \in \tilde{S}_n \setminus \tilde{A}_n$. Up to now, we have

$$t_{\lambda^\pm} t_{\mu^\pm_*} \sim_{\tilde{A}_n} \begin{cases} z^e t_{\nu_\geqslant} & \text{if } o \text{ is even,} \\ z^e t^t_{\nu_\geqslant} & \text{if } o \text{ is odd;} \end{cases}$$
$$t_{\lambda^\pm} t_{\mu^\mp_*} \sim_{\tilde{A}_n} \begin{cases} z^e t^t_{\nu_\geqslant} & \text{if } o \text{ is even,} \\ z^e t_{\nu_\geqslant} & \text{if } o \text{ is odd.} \end{cases}$$

Let f be the number of even parts of ν and $t_{\nu_\geqslant} = t(\nu_1, \ell_1) \cdots t(\nu_r, \ell_r)$. Suppose first that $f > 0$ and let ν_i be an even part of ν. Then

$$t_{\nu_\geqslant} = t(\nu_1, \ell_1) \cdots t(\nu_i, \ell_i)^{t(\nu_i, \ell_i)} \cdots t(\nu_r, \ell_r) = z^{f-1} t_{\nu_\geqslant}^{t(\nu_i, \ell_i)}$$

because for every $j \in \{1, \ldots, r\} \setminus \{i\}$ we have $t(\nu_j, \ell_j)^{t(\nu_i, \ell_i)} = z t(\nu_j, \ell_j)$ if ν_j is even and $t(\nu_j, \ell_j)^{t(\nu_i, \ell_i)} = t(\nu_j, \ell_j)$ if ν_j is odd (see (5.1.1) and (5.1.2)). Therefore, we have

$$z^e t^t_{\nu_\geqslant} = z^{e+f-1} t_{\nu_\geqslant}^{t(\nu_i, \ell_i) t} \sim_{\tilde{A}_n} z^{e+f-1} t_{\nu_\geqslant}.$$

Suppose next that $f = 0$ and thus $\nu_\geqslant \in \mathcal{O}(n)$. If $\nu_\geqslant \notin \mathcal{D}(n)$, then there are two equal, odd parts $\nu_i = \nu_j$ $(i \neq j)$. Then $t(\nu_m, \ell_m) = t_{\ell_m+1} \cdots t_{\ell_m+\nu_m-1}$ for $m \in \{i, j\}$ and $s := t_{\ell_i+1, \ell_j+1} \cdots t_{\ell_i+\nu_i, \ell_j+\nu_j} \in \tilde{S}_n \setminus \tilde{A}_n$ makes $t(\nu_i, \ell_i)^s = t(\nu_j, \ell_j)$, whereas it centralizes each $t(\nu_m, \ell_m)$ for $m \in \{1, \ldots, r\} \setminus \{i, j\}$, see (5.1.1). Moreover, as $\nu_\geqslant \in \mathcal{O}(n)$ the factors $t(\nu_a, \ell_a)$ and $t(\nu_b, \ell_b)$ commute for all $a, b \in \{1, \ldots, r\}$, so that s centralizes $t_{\nu_\geqslant}$. Thus

$$z^e t^t_{\nu_\geqslant} = z^e t^{st}_{\nu_\geqslant} \sim_{\tilde{A}_n} z^e t_{\nu_\geqslant}.$$

Finally, if $f = 0$ and there are no two parts of equal length, then $\nu_\geqslant \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$. But then $z^e t^t_{\nu^+_\geqslant} \sim_{\tilde{A}_n} z^e t_{\nu^-_\geqslant}$ as $t \in \tilde{S}_n \setminus \tilde{A}_n$. We summarize: if $\nu_\geqslant \notin \mathcal{O}(n) \cup \mathcal{D}^+(n)$, then $t_{\lambda^\pm} t_{\mu^\pm_*} \sim_{\tilde{A}_n} t_{\lambda^\pm} t_{\mu^\mp_*} \sim_{\tilde{A}_n} t_{\nu_\geqslant}$. Otherwise, if $\nu_\geqslant \in \mathcal{O}(n) \cup \mathcal{D}^+(n)$, then

(A) the conjugacy class of $x \in \{t_{\lambda^+}t_{\mu_*^+}, t_{\lambda^-}t_{\mu_*^-}\}$ in $\tilde{A}_n$ is determined as follows:

(1) if o is even, then

(a) if $\nu_{\geqslant} \notin \mathcal{O}(n) \cap \mathcal{D}^+(n)$, then $x \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}}$;

(b) otherwise, $x \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}^+}$;

(2) if o is odd, then

(a) if $f > 0$, then $x \sim_{\tilde{A}_n} z^{e+f-1} t_{\nu_{\geqslant}}$;

(b) if $f = 0$, that is $\nu_{\geqslant} \in \mathcal{O}(n)$, then

(i) if $\nu_{\geqslant} \notin \mathcal{D}^+(n)$, then $x \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}}$;

(ii) otherwise, $x \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}^-}$;

(B) the conjugacy class of $y \in \{t_{\lambda^+}t_{\mu_*^-}, t_{\lambda^-}t_{\mu_*^+}\}$ in $\tilde{A}_n$ is determined as follows:

(1) if o is odd, then

(a) if $\nu_{\geqslant} \notin \mathcal{O}(n) \cap \mathcal{D}^+(n)$, then $y \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}}$;

(b) otherwise, $y \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}^+}$;

(2) if o is even, then

(a) if $f > 0$, then $y \sim_{\tilde{A}_n} z^{e+f-1} t_{\nu_{\geqslant}}$;

(b) if $f = 0$, that is $\nu_{\geqslant} \in \mathcal{O}(n)$, then

(i) if $\nu_{\geqslant} \notin \mathcal{D}^+(n)$, then $y \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}}$;

(ii) otherwise, $y \sim_{\tilde{A}_n} z^e t_{\nu_{\geqslant}^-}$. ∎

5.5 Some known results on decomposition numbers

Let $X \leqslant \tilde{S}_n$ be such that z is contained in the commutator subgroup of X, that is, X does not split over $\langle z \rangle$. Then $\varphi \in \mathrm{Irr}(X) \cup \mathrm{IBr}(X)$ is a spin character if $\varphi(z) = -\varphi(1)$. It can be seen by looking at the central characters ω_χ for $\chi \in \mathrm{Irr}(X)$ that for a block B of X either every element of $\mathrm{Irr}(B)$ is a spin character or none. The analogue statement holds for $\mathrm{IBr}(B)$ because the decomposition map is surjective. We call B a spin p-block if B contains any spin character.

(5.5.1) Theorem (Olsson [60]). Let B be a spin p-block of $\tilde{S}_n$ of positive defect.

(1) Either $\sigma^a = \sigma$ for all $\sigma \in \mathrm{IBr}(B)$, or $\sigma^a \neq \sigma$ for all $\sigma \in \mathrm{IBr}(B)$.

(2) B covers a unique block b of $\tilde{A}_n$. In particular, either $\lambda^c \neq \lambda$ for all $\lambda \in \mathrm{IBr}(b)$, or $\lambda^c = \lambda$ for all $\lambda \in \mathrm{IBr}(b)$. ∎

Note that the preceding result can be recovered from the block description of the superalgebra $f\tilde{S}_n$, see (4.2.3). Let B be a spin p-block of $\tilde{S}_n$ of positive defect that covers the spin p-block b of $\tilde{A}_n$. It follows from (5.5.1) and (1.6) that

(i) $\mathrm{IBr}(B){\downarrow_{\tilde{A}_n}} = \mathrm{IBr}(b)$ if all Brauer characters of B are non-self-associate, and

(ii) $\mathrm{IBr}(B) = \mathrm{IBr}(b){\uparrow^{\tilde{S}_n}}$ if all Brauer characters of b are non-self-conjugate.

These relations affect the decomposition numbers as follows.

(5.5.2) Corollary. Let $\chi \in \mathrm{Irr}(B)$ and $\psi \in \mathrm{Irr}(b)$.

(1) Assume that $\sigma \in \mathrm{IBr}(B)$ such that $\sigma^a \neq \sigma$ and $\sigma{\downarrow_{\tilde{A}_n}} = \lambda \in \mathrm{IBr}(b)$.
   - (a) If $\chi \neq \chi^a$ and $\chi{\downarrow_{\tilde{A}_n}} = \psi$, then $d_{\psi\lambda} = d_{\chi\sigma} + d_{\chi^a\sigma}$.
   - (b) If $\chi = \chi^a$ and $\chi{\downarrow_{\tilde{A}_n}} = \psi + \psi^c$, then $d_{\psi\lambda} = d_{\chi\sigma} = d_{\psi^c\lambda}$.

(2) Assume that $\lambda \in \mathrm{IBr}(b)$ such that $\lambda^c \neq \lambda$ and $\lambda{\uparrow^{\tilde{S}_n}} = \sigma \in \mathrm{IBr}(B)$.
   - (a) If $\psi = \psi^c$ and $\psi{\uparrow^{\tilde{S}_n}} = \chi + \chi^a$, then $d_{\chi\sigma} = d_{\psi\lambda} = d_{\chi^a\sigma}$.
   - (b) If $\psi \neq \psi^c$ and $\psi{\uparrow^{\tilde{S}_n}} = \chi$, then $d_{\chi\sigma} = d_{\psi\lambda} + d_{\psi^c\lambda}$.

Proof. Recall that $d_{\chi\sigma} = d_{\chi^a\sigma^a}$ and $d_{\psi\lambda} = d_{\psi^c\lambda^c}$, thus $d_{\chi\sigma^a} = d_{\chi^a\sigma}$ and $d_{\psi\lambda^c} = d_{\psi^c\lambda}$ for all $\chi, \sigma \in B$, $\psi, \lambda \in b$. For part (1), let $\mathcal{J}$ be a complete set of representatives of the classes of associate characters in $\mathrm{IBr}(B)$. Recall $\sigma{\downarrow_{\tilde{A}_n}} = \lambda$. Then (a) follows from

$$\psi|_{p'} = (\chi|_{p'}){\downarrow_{\tilde{A}_n}} = \sum_{\sigma\in\mathcal{J}} (d_{\chi\sigma}\sigma{\downarrow_{\tilde{A}_n}} + d_{\chi\sigma^a}\sigma^a{\downarrow_{\tilde{A}_n}}) = \sum_{\sigma\in\mathcal{J}} (d_{\chi\sigma} + d_{\chi^a\sigma})\sigma{\downarrow_{\tilde{A}_n}} .$$

Similarly, in (b) we have

$$\sum_{\lambda\in\mathrm{IBr}(b)} (d_{\psi\lambda} + d_{\psi^c\lambda})\lambda = \psi|_{p'} + (\psi|_{p'})^c = (\chi|_{p'}){\downarrow_{\tilde{A}_n}} = \sum_{\sigma\in\mathcal{J}} (d_{\chi\sigma} + d_{\chi\sigma^a})\sigma{\downarrow_{\tilde{A}_n}} .$$

As $\chi = \chi^a$ and $\lambda = \lambda^c$, we have $d_{\chi\sigma} = d_{\chi\sigma^a}$ and $d_{\psi\lambda} = d_{\psi^c\lambda}$, hence the claim follows. Part (2) is proven analogously. ∎

In practice, however, we approach the problem of computing decomposition numbers the other way around. This is due to the experience that the absence of pairs of conjugate respectively associate Brauer characters eases the calculation. In (i) above, knowing $\mathrm{IBr}(b)$ provides valuable information for the computation of $\mathrm{IBr}(B)$. Likewise, in (ii) we would usually compute $\mathrm{IBr}(B)$ first and then proceed onwards to $\mathrm{IBr}(b)$.
The following theorem describes the reduction of the basic spin characters of $\tilde{S}_n$.

(5.5.3) Theorem (Wales[69]). Let p be a rational prime and $n \geqslant 4$.

(1) Assume that $p \nmid n$.

  (a) If n is even, then $\sigma(n) := \langle\langle n,+\rangle\rangle|_{p'}$ and $\sigma(n)^a = \langle\langle n,-\rangle\rangle|_{p'}$ are irreducible p-modular characters of $\tilde{S}_n$ that are associate.

  (b) If n is odd, then $\sigma(n) := \langle\langle n\rangle\rangle|_{p'}$ is an irreducible p-modular character of $\tilde{S}_n$ that is self-associate.

(2) Assume that $p \mid n$.

  (a) If n is even, then $\sigma(n) := \langle\langle n,+\rangle\rangle|_{p'}$ is an irreducible p-modular character of $\tilde{S}_n$ that is self-associate and equal to $\langle\langle n,-\rangle\rangle|_{p'}$.

  (b) If n is odd, then $\langle\langle n\rangle\rangle|_{p'} = \sigma(n)+\sigma(n)^a$ where $\sigma(n)$ is an irreducible p-modular character of $\tilde{S}_n$ that is non-self-associate. ∎

In addition, Wales worked out the p-modular reduction of the $\tilde{S}_n$-characters labelled by $(n-1,1)$ that are sometimes called the *second* basic spin characters. We note that the identification of the representation Z_n from [69] with these characters follows from the ordinary branching rules derived from Morris' recursion formula [51] (see [21], (4.2.7)) and the property that Z_n appears in the basic spin character of $\tilde{S}_{n-1}$ induced up to $\tilde{S}_n$.

(5.5.4) Theorem (Wales [69]). Let p be an odd rational prime and $n \geqslant 6$.

(1) Assume that $p \nmid n(n-1)$.

(a) If n is even, then $\sigma(n-1,1) := \langle\!\langle n-1,1\rangle\!\rangle|_{p'}$ is an irreducible p-modular character of $\tilde{S}_n$ that is self-associate.

(b) If n is odd, then $\sigma(n-1,1) := \langle\!\langle n-1,1,+\rangle\!\rangle|_{p'}$ and $\sigma(n-1,1)^a = \langle\!\langle n-1,1,-\rangle\!\rangle|_{p'}$ are irreducible p-modular characters of $\tilde{S}_n$ that are associate.

(2) Assume that $p \mid n$.

(a) If n is even, then $\sigma(n-1,1) := \langle\!\langle n-1,1\rangle\!\rangle|_{p'} - \sigma(n)$ is an irreducible p-modular character of $\tilde{S}_n$ that is self-associate.

(b) If n is odd, then $\sigma(n-1,1) := \langle\!\langle n-1,1,+\rangle\!\rangle|_{p'} - \sigma(n)$ and $\sigma(n-1,1)^a = \langle\!\langle n-1,1,-\rangle\!\rangle|_{p'} - \sigma(n)^a$ are irreducible p-modular characters of $\tilde{S}_n$ that are associate.

(3) Assume that $p \mid (n-1)$.

(a) If n is even, then $\langle\!\langle n-1,1\rangle\!\rangle|_{p'} - \sigma(n) - \sigma(n)^a = \sigma(n-1,1) + \sigma(n-1,1)^a$ where $\sigma(n-1,1)$ is an irreducible p-modular character of $\tilde{S}_n$ that is non-self-associate.

(b) If n is odd, then $\sigma(n-1,1) := \langle\!\langle n-1,1,+\rangle\!\rangle|_{p'} - \sigma(n)$ is an irreducible p-modular character of $\tilde{S}_n$ that is self-associate and equal to $\langle\!\langle n-1,1,-\rangle\!\rangle|_{p'} - \sigma(n)$. ∎

6 Basic spin representations

In [67], Schur constructs a faithful irreducible matrix representation X_n of $\tilde{S}_n$ over $\mathbb{C}$, the *basic* representation whose character is indexed by (n). Morris [49] added the term "spin" describing the origin of X_n from the spin representation of the orthogonal group. By (5.5.3), the p-modular reduction of a basic spin representation is irreducible unless n is odd and divisible by p. In the latter case, there are exactly two distinct and associate composition factors of equal degree. In this way, a composition factor of a p-modular reduction of an ordinary basic spin representation is said to be a (p-modular) basic spin representation. Likewise names are given to the associate representation X_n^a (which is equivalent to X_n only if n is odd) and its p-modular constituents as well as the constituents of the restriction of X_n to $\tilde{A}_n$. The degree of a basic spin representation of $\tilde{S}_n$ over a splitting field F of characteristic $p \geqslant 0$ is given by

$$\delta(\tilde{S}_n) = \begin{cases} 2^{k-1} & \text{if } n = 2k, \\ 2^{k-1} & \text{if } n = 2k+1 \text{ and } p \mid n, \\ 2^k & \text{if } n = 2k+1 \text{ and } p \nmid n. \end{cases}$$

It is worth pointing out that the basic spin representations have minimal degree among the faithful representations of $\tilde{S}_n$ and $\tilde{A}_n$, see [34, 35]. A different characterisation ([69]) states that the basic spin representations are the only faithful irreducible representations of $\tilde{S}_n$ (or S_n if $p = 2$) under which an element of order 3 projecting to a 3-cycle has the quadratic minimal polynomial $x^2 + x + 1$. This characterisation allows a recursive construction of a basic spin representation of $\tilde{S}_n$ for every given pair (n, p). This is described in detail in [45]. We reproduce the recipe of construction here.

We identify a representation T of $\tilde{S}_n$ of degree d over a field F with the sequence $(t_1^T, \ldots, t_{n-1}^T) \in \mathrm{GL}_d(F)^{n-1}$. By I we denote the identity matrix and by 0 we denote the zero matrix, both of suitable degree over F. For a given matrix M over F and any integer

μ, we write μM instead of $(\mu \cdot 1_F)M$. First of all,

$$\left(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right)$$

is a basic "spin" representation of $\tilde{S}_4$ over GF(2),

$$\left(\begin{bmatrix} \zeta^2 & \\ & \zeta^6 \end{bmatrix}, \begin{bmatrix} \zeta^2 & \zeta^4 \\ & \zeta^6 \end{bmatrix}, \begin{bmatrix} & 1 \\ \zeta^4 & \end{bmatrix} \right)$$

is a basic spin representation of $\tilde{S}_4$ over GF(9) with primitive element ζ, and

$$\left(\begin{bmatrix} \omega & -1 \\ & -\omega \end{bmatrix}, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \begin{bmatrix} -\alpha\omega & 2\alpha - 1 \\ 2\alpha & \alpha\omega \end{bmatrix} \right)$$

is a basic spin representation of $\tilde{S}_4$ over $GF(p^2)$ for $p > 3$ with primitive element ζ, $\omega = \zeta^{(p^2-1)/4}$, and $\alpha = 3^{-1}(1 + \omega\sqrt{2})$.

($\mathfrak{S}$) Construction. Let $T = (T_1, \ldots, T_{n-2})$ be a basic spin representation of $\tilde{S}_{n-1}$ over a field F of characteristic $p \geqslant 0$. Then $\mathfrak{S}(T) = (T_1^\uparrow, \ldots, T_{n-1}^\uparrow)$ defined by

$$T_i^\uparrow = \begin{bmatrix} T_i & \\ & -T_i \end{bmatrix} \ (1 \leqslant i \leqslant n-3), \ T_{n-2}^\uparrow = \begin{bmatrix} T_{n-2} & -I \\ & -T_{n-2} \end{bmatrix}, \quad T_{n-1}^\uparrow = \begin{bmatrix} & I \\ -I & \end{bmatrix}$$

is a basic spin representation of $\tilde{S}_n$ over F if either n is odd and $p \nmid n$, or n is even and $p \mid (n-1)$. Otherwise, $\mathfrak{S}(T)$ has exactly two composition factors and these are basic spin. ∎

($\mathfrak{S}_{\mathrm{I}}$) Construction. Let $T = (T_1, \ldots, T_{n-2})$ be a basic spin representation of $\tilde{S}_{n-1}$ over a field F of characteristic p. Define

$$T_{n-1} = \left\{ \begin{array}{ll} J & \text{if } p > 2 \\ J & \text{if } p = 2, \ n \equiv 0 \bmod 4 \\ J + I & \text{if } p = 2, \ n \equiv 2 \bmod 4 \end{array} \right\} \text{ where } J = \sum_{k=1}^{n-2} kT_k.$$

If $p \mid n$, then $\mathfrak{S}_{\mathrm{I}}(T) = (T_1, \ldots, T_{n-2}, T_{n-1})$ is a basic spin representation of $\tilde{S}_n$ over F. ∎

($\mathfrak{S}_{\mathrm{II}}$) Construction. Provided $n \geqslant 6$ is even, let $T = (T_1, \ldots, T_{n-3})$ be a basic spin representation of $\tilde{S}_{n-2}$ over a field F of characteristic $p \geqslant 0$ such that $p \nmid n(n-1)(n-2)$.

Then

$$T_{n-1} = \begin{bmatrix} -\alpha J & (\beta - 1)I \\ \beta I & \alpha J \end{bmatrix}$$

where

$$J = \sum_{k=1}^{n-3} kT_k, \quad \alpha = (n-1)^{-1}\left(1 \pm \sqrt{-n(n-2)^{-1}}\right) \text{ and } \beta = (n-2)\alpha,$$

extends $\mathfrak{S}(T)$ to a basic spin representation of $\tilde{S}_n$ over $F(\alpha)$. ∎

($\mathfrak{S}_{III}$) Construction. Let $n \geqslant 8$ be even. If $T = (T_1, \ldots, T_{n-5})$ is a basic spin representation of $\tilde{S}_{n-4}$ over a field F of characteristic $p \geqslant 3$ such that $p \mid (n-2)$, define

$$T_{n-1} = \begin{bmatrix} J & -I \\ & -J \end{bmatrix} \quad \text{where} \quad J = \pm\sqrt{-1}\begin{bmatrix} \sum_{k=1}^{n-5} kT_k & 2I \\ 2I & -\sum_{k=1}^{n-5} kT_k \end{bmatrix}.$$

The matrix T_{n-1} extends the basic spin representation $\mathfrak{S}\,\mathfrak{S}_I\,\mathfrak{S}(T)$ of $\tilde{S}_{n-1}$ to $\tilde{S}_n$. The field of definition is $F(\sqrt{-1})$. ∎

(6.1) Theorem ([45]). Let $n \geqslant 4$. For $F = \mathbb{C}$, GF(2), or GF(p^2) with $p \geqslant 3$, iterative use of the constructions $\mathfrak{S}$, $\mathfrak{S}_I$, $\mathfrak{S}_{II}$, or $\mathfrak{S}_{III}$, as described in Table 6.1, provides a basic spin representation of $\tilde{S}_n$ over F. ∎

Table 6.1: Basic spin constructions over F

F		n odd	n even
$\mathbb{C}$		$\mathfrak{S}$	$\mathfrak{S}_{II}$
GF(2)		$\mathfrak{S}$	$\mathfrak{S}_I$
GF(p^2)	$p \nmid n(n-1)(n-2)$	$\mathfrak{S}$	$\mathfrak{S}_{II}$
($p \geqslant 3$)	$p \mid n$	$\mathfrak{S}_I$	$\mathfrak{S}_I$
	$p \mid (n-1)$	$\mathfrak{S}$	$\mathfrak{S}$
	$p \mid (n-2)$	$\mathfrak{S}$	$\mathfrak{S}_{III}$

In addition to Section 4, the basic spin $f\tilde{S}_n$-supermodule $D(\omega_n)$, see [32, (22.3.3)], is labelled by

$$\omega_n := \begin{cases} (p^a, b) = (p, \ldots, p, b) & \text{if } n = ap + b \text{ with } 0 < b < p, \\ (p^{a-1}, p-1, 1) & \text{if } n = ap. \end{cases}$$

Note that $\omega_n = (n)$ if $p \geqslant n$. This can be seen as follows by interpreting Wales' results in terms of supermodules.#[1] Let D_n denote a basic spin $f\tilde{S}_n$-supermodule. By induction, we may assume that $D_{n-1} = D(\omega_{n-1})$ for $n \geqslant 5$. Using the branching rules, see 4.2, we have

$$X((n))\downarrow_{\tilde{S}_{n-1}} \cong \begin{cases} X((n-1)) & \text{if } n-1 \text{ is even}, \\ 2 \cdot X((n-1)) & \text{if } n-1 \text{ is odd} \end{cases}$$

for the basic spin $\mathcal{F}\tilde{S}_n$-supermodule $X((n))$. Therefore, we can deduce that the restriction of D_n to $\tilde{S}_{n-1}$ has only one constituent and this is isomorphic to D_{n-1} (with multiplicity one unless n is odd and divisible by p, in which case $D_n\downarrow_{\tilde{S}_{n-1}}$ contains D_{n-1} twice). In particular, the socle of $D(\omega_{n-1})\uparrow^{\tilde{S}_n}$ has a constituent that is isomorphic to D_n. Now $\mathrm{soc}(D(\omega_{n-1})\uparrow^{\tilde{S}_n})$ is under control by (4.2.9). A case by case analysis in terms of the combinatorial notions from Section 4 is needed to determine the cogood nodes that can be added to ω_{n-1}, showing that indeed $D_n = D(\omega_n)$.

[1] Here we need to consider p-modular reduction for supermodules. Suppose that $f = \mathcal{O}/\wp$ where $\wp$ is the maximal ideal of $\mathcal{O}$ containing p. If V is an $\mathcal{F}\tilde{S}_n$-supermodule, there is an "$\mathcal{O}$-superform" $V_{\mathcal{O}}$, that is, an $\mathcal{F}\tilde{S}_n$-invariant $\mathcal{O}$-subsuperspace of V with $V_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F} = V$. Then we may speak of the p-modular reduction $\overline{V} := V_{\mathcal{O}} \otimes_{\mathcal{O}} f$ of V.

7 The 3-modular spin characters of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

Preliminaries

From here on we calculate the p-modular spin characters of $\tilde{S}_n$ and $\tilde{A}_n$ for $n \in \{14, \dots, 18\}$ and $p \in \{3, 5, 7\}$. The underlying ordinary character tables of $\tilde{S}_n$ and $\tilde{A}_n$ are always obtained by the GAP-functions (see [56])

```
CharacterTable( "DoubleCoverSymmetric", n ),
CharacterTable( "DoubleCoverAlternating", n ).
```

At the beginning of each section we collect some information on the relevant spin blocks. For a specific p-block B_i its index i is the position in the list output by the GAP-function `PrimeBlocks()`. Then we give its block constants $d(B_i)$, $k(B_i)$, and $l(B_i)$, as well as its p-bar core $c(B_i)$ and its p-weight $w(B_i)$. Note that the latter two coincide for a spin p-block B_i of $\tilde{S}_n$ and the unique p-block b_i of $\tilde{A}_n$ that is covered by B_i (see (5.5.1)). We use capital letters B_i to denote blocks of $\tilde{S}_n$, in contrast to small letters b_i for blocks of $\tilde{A}_n$. Also, to ease distinction between characters of $\tilde{S}_n$ and characters of $\tilde{A}_n$, we introduce some more notation: a character $\|\nu\|$ in the sense of 5.3 if prefixed by 'λ' if $\|\nu\| \in \mathrm{Irr}(X) \cup \mathrm{IBr}(X)$ and by 'Λ' if $\|\nu\| \in \mathrm{IPr}(X)$ for some $X \leqslant \tilde{A}_n$. Likewise, we prefix by 'σ' and 'Σ' if $X \leqslant \tilde{S}_n$ but $X \nleqslant \tilde{A}_n$. In this way,

σ_i	denotes a Brauer character of $\tilde{S}_n$,
Σ_i	denotes a projective character of $\tilde{S}_n$,
λ_i	denotes a Brauer character of $\tilde{A}_n$,
Λ_i	denotes a projective character of $\tilde{A}_n$.

Irreducible or projective indecomposable characters are often underlined in due course. Moreover, if for example $\underline{\sigma} \in \mathrm{IBr}(B_i)$ or $\underline{\Sigma} \in \mathrm{IPr}(B_i)$ are known from a subsection about B_i, in a different subsection we will refer to $\underline{\sigma}$ or $\underline{\Sigma}$ as $\underline{\sigma}(B_i)$ or $\underline{\Sigma}(B_i)$, respectively. Finally, the elements of the underlying special basic sets are marked by a '$\star$' in the decomposition matrices.

The block constants of the spin 3-blocks of $\tilde{S}_{14}$ and $\tilde{A}_{14}$ are collected in Table 7.1.

Table 7.1: Spin 3-blocks of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
7	21	10	5	(2)	4	5	21	5
8	12	3	4	$(4,1)$	3	6	12	6

By (5.5.3), the 3-modular basic spin characters of $\tilde{S}_{14}$ are $\sigma(14) := \sigma\langle\langle 14,+\rangle\rangle|_{3'}$ and $\sigma(14)^a = \sigma\langle\langle 14,-\rangle\rangle|_{3'}$. The 3-modular basic spin character of $\tilde{A}_{14}$ is $\lambda(14) := \sigma(14)\downarrow_{\tilde{A}_{14}}$. Moreover, by (5.5.4), $\sigma(13,1) := \sigma\langle\langle 13,1\rangle\rangle|_{3'}$ is an irreducible Brauer character of $\tilde{S}_{14}$. As $\sigma\langle\langle 13,1\rangle\rangle\downarrow_{\tilde{A}_{14}} = \lambda\langle\langle 13,1,+\rangle\rangle + \lambda\langle\langle 13,1,-\rangle\rangle$, we can read off two irreducible Brauer characters of $\tilde{A}_{14}$, $\lambda(13,1) := \lambda\langle\langle 13,1,+\rangle\rangle|_{3'}$ and $\lambda(13,1)^c = \lambda\langle\langle 13,1,-\rangle\rangle|_{3'}$.

7.1 The block B_8

Let σ_i and Σ_i be defined by Table 7.2.

Table 7.2: Some characters for B_8

i	σ_i	$\deg\sigma_i$	Σ_i	$\deg\Sigma_i$
1	$\sigma(13,1)$	768	$(\sigma(14)\otimes\Sigma\langle 10,2,1^2\rangle)|_{B_8}$	1259712
2	$\sigma[\![5,4,2:2,1,+]\!]\uparrow^{\tilde{S}_{14}}|_{B_8}$	30912	$\Sigma[\![7,4,2]\!]\uparrow^{\tilde{S}_{14}}|_{B_8}$	388800
3	$(\sigma(14)\otimes\sigma[10,4])|_{B_8}$	26688	$\Sigma[\![3^4,1]\!]\uparrow^{\tilde{S}_{14}}|_{B_8}$	4385664

Then $\{\sigma_1,\sigma_2,\sigma_3\}$ is a basic set of Brauer characters and $\{\Sigma_1,\Sigma_2,\Sigma_3\}$ is a basic set of projective characters for B_8 since the matrix of mutual scalar products is invertible over $\mathbb{Z}$:

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3
σ_1	1	.	3
σ_2	.	1	.
σ_3	.	.	1

We see that $\underline{\sigma}_2 := \sigma_2 = \Sigma_2^*$ and $\underline{\sigma}_3 := \sigma_3 = \Sigma_3^*$ are Brauer atoms and therefore irreducible, see (2.8). In particular, as $\underline{\sigma}_1 := \sigma_1$ is irreducible too, it follows that the projective indecomposable characters are Σ_1, Σ_2, and $\Sigma_3 - 3\Sigma_1$. The resulting decomposition matrix of B_8 is Table 7.14.

7.2 The block b_6

Table 7.1 shows that b_6 is covered by B_8, see (5.5.1). All irreducible Brauer characters of B_8 are self-associate, hence all irreducible Brauer characters of b_6 are non-self-conjugate (see (1.6)). The degrees of the irreducible Brauer characters of b_6 are 384, 15456, and 13344. We already know the pair of irreducibles of degree 384, $\underline{\lambda}_1 := \lambda(13,1)$ and its conjugate character $\underline{\lambda}_1^c$. Let $\lambda_{2a} := \lambda[\![6,4,2,1,-]\!]\uparrow^{\tilde{A}_{14}}|_{b_6}$ and $\lambda_{2b} := \lambda[\![6,4,2,1,+]\!]\uparrow^{\tilde{A}_{14}}|_{b_6}$. Then $\underline{\sigma}_2\downarrow_{\tilde{A}_{14}} = \lambda_{2a} + \lambda_{2b}$ implies that $\underline{\lambda}_2 := \lambda_{2a}$ and $\underline{\lambda}_2^c = \lambda_{2b}$ are irreducible. Their degree is 15456. For $\lambda_3 := \lambda\langle\!\langle 10,4,+\rangle\!\rangle|_{3'}$ with $\deg \lambda_3 = 13728$, we have

$$\lambda_3\uparrow^{\tilde{S}_{14}} = \underline{\sigma}_1 + \underline{\sigma}_3,$$

hence either $\underline{\lambda}_1$ or $\underline{\lambda}_1^c$ is contained in λ_3. In particular, either $\{\lambda_3 - \underline{\lambda}_1, \lambda_3^c - \underline{\lambda}_1^c\}$ or $\{\lambda_3^c - \underline{\lambda}_1, \lambda_3 - \underline{\lambda}_1^c\}$ is the missing pair of irreducible Brauer characters of b_6. These two pairs differ as follows (in ATLAS notation [12]):

	$t_{(10,4)}$	$zt_{(10,4)}$	$t_{(13,1)^+}$	$zt_{(13,1)^+}$	$t_{(13,1)^-}$	$zt_{(13,1)^-}$
$\lambda_3 - \underline{\lambda}_1$	A	$-A$	B	$-B$	$*B$	$-*B$
$\lambda_3^c - \underline{\lambda}_1^c$	$-A$	A	$*B$	$-*B$	B	$-B$
$\lambda_3 - \underline{\lambda}_1^c$	A	$-A$	$*B$	$-*B$	B	$-B$
$\lambda_3^c - \underline{\lambda}_1$	$-A$	A	B	$-B$	$*B$	$-*B$

$A = -r10$, $B = 1 + b13$

We use the MeatAxe (see Section 3) to construct a 13344-dimensional irreducible representation $13344a^-$ of $\tilde{A}_{14}$ over GF(3) that affords one of the two Brauer characters in question: first, we build a basic spin representation of $\tilde{S}_{14}$ over GF(9) as described in Section 6. Denote its restriction to $\tilde{A}_{14}$ by $64a^-$. We choose $t_{(10,4)}$ and $t_{(13,1)^+}$ as generators a and b for $\tilde{A}_{14}$, respectively. Then

- realize $64a^-$ over GF(3) by means of the word $w := a + b + ab + ab^2$, see (3.3)
- find $560a^+$ by chopping the sequence $(64a^- \otimes 64a^-, 13a^+ \otimes 78a^+, 13a^+ \otimes 273a^+)$

- build 35840a $:= 64a^- \otimes 560a^+$ and calculate the row nullspace N of w. Spinning up the second vector of N under the action of 35840a yields a submodule 9152a with corresponding quotient 26688a that has $13344a^-$ as a constituent.

Comparing

$$\overline{A} \equiv 2 \bmod 3, \quad \overline{B} \equiv 1 \bmod 3, \quad \overline{*B} \equiv 0 \bmod 3$$

with the traces $\mathrm{tr}_{13344a^-}(a) \equiv 2 \bmod 3$ and $\mathrm{tr}_{13344a^-}(b) \equiv 1 \bmod 3$ identifies $\underline{\lambda}_3 := \lambda_3 - \underline{\lambda}_1$ as the Brauer character afforded by $13344a^-$. Thus $\underline{\lambda}_3$ and $\underline{\lambda}_3^c = \lambda_3^c - \underline{\lambda}_1^c$ complete $\mathrm{IBr}(b_6)$. The decomposition matrix of b_6 is Table 7.15.

7.3 The block b_5

The block b_5 contains five irreducible Brauer characters, each of which has two associate extensions in B_7. Let the Brauer characters λ_i and projective characters Λ_i be defined by Table 7.3.

Table 7.3: Some characters for b_5

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(14)$	64	$\Lambda[\![3^4, 1, -]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_5}$	2379456
2	$\lambda[\![4, 3^2, 2, 1, -]\!]\uparrow^{\tilde{A}_{14}}$	2016	$\Lambda[\![5, 3^2, 2]\!]\uparrow^{\tilde{A}_{14}}$	653184
3	$\lambda\langle\!\langle 5, 4, 3, 2, +\rangle\!\rangle\vert_{3'}$	4576	$\Lambda[\![5, 4, 3, 1, +]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_5}$	909792
4	$\lambda[\![6, 4, 2, 1, -]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_5}$	13664	$\Lambda[\![6, 5, 2]\!]\uparrow^{\tilde{A}_{14}}$	762048
5	$\lambda\langle\!\langle 8, 5, 1\rangle\!\rangle\vert_{3'}$	68992	$(\Lambda[\![7, 4, 2, -]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_5})/2$	256608
6	$\lambda[\![3^2, 1, - : 4, 2, 1, -]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_5}$	61920		

Recall that $\underline{\lambda}_1 := \lambda_1$ is irreducible. Then $\{\underline{\lambda}_1, \lambda_2, \ldots, \lambda_5\}$ is a basic set of Brauer characters and $\{\Lambda_1, \ldots, \Lambda_5\}$ is a basic set of projective characters for b_5. The matrix of mutual scalar products (Table 7.4) shows that $\underline{\lambda}_2 := \lambda_2 = \Lambda_2^*$, $\underline{\lambda}_3 := \lambda_3 = \Lambda_3^*$, and $\underline{\lambda}_4 := \lambda_4 = \Lambda_4^*$ are Brauer atoms, thus irreducible by (2.8). Moreover, $\underline{\Lambda}_2 := \Lambda_2$ and $\underline{\Lambda}_5 := \Lambda_5$ are projective atoms and therefore indecomposable. The relation

$$\lambda_6 = \underline{\lambda}_2 + \underline{\lambda}_3 - \underline{\lambda}_4 + \lambda_5$$

shows that λ_5 contains $\underline{\lambda}_4$. Thus $\underline{\Lambda}_4 := \Lambda_4$ is indecomposable. Exchanging λ_5 with

Table 7.4: Mutual scalar products (b_5)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
$\underline{\lambda}_1$	1	.	.	.	.
λ_2	.	1	.	.	.
λ_3	.	.	1	.	.
λ_4	.	.	.	1	.
λ_5	4	.	1	1	1

$\lambda_{5a} := \lambda_5 - \underline{\lambda}_4$ yields an improved basic set with mutual scalar products given in Table 7.5. By now, we have $\deg \underline{\lambda}_1 = 64$, $\deg \underline{\lambda}_2 = 2016$, $\deg \underline{\lambda}_3 = 4576$, $\deg \underline{\lambda}_4 = 13664$, and

Table 7.5: Mutual scalar products (b_5)

$\langle\cdot,\cdot\rangle$	Λ_1	$\underline{\Lambda}_2$	Λ_3	$\underline{\Lambda}_4$	$\underline{\Lambda}_5$
$\underline{\lambda}_1$	1	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.
$\underline{\lambda}_4$	.	.	.	1	.
λ_{5a}	4	.	1	.	1

$\deg \lambda_{5a} = 55328$. Hence the missing irreducible Brauer character $\underline{\lambda}_5$ is of the form

$$\underline{\lambda}_5 = \lambda_{5a} - x\underline{\lambda}_1 - y\underline{\lambda}_3$$

for suitable integers x, y with $0 \leqslant x \leqslant 4$ and $0 \leqslant y \leqslant 1$; in particular, $\underline{\lambda}_5$ is uniquely determined by its degree, which is at least 50496.

As described in Section 3, we condense the tensor product $V := 64a^- \otimes 3562a^+$ with respect to the $3'$-subgroup $K := \langle t_{(7,1^7)}, t_{(7,7)} \rangle \cong 7^2$, where $64a^-$ is the basic spin representation of $\tilde{A}_{14}$ and $3562a^+$ is an irreducible representation which affords $\lambda[8,5,1]$, both realized over $F := GF(3)$. It can be seen from the Brauer characters of the basic set that every irreducible module of b_5 can be realized over F. Here $3562a^+$ is constructed by chopping the sequence $(13a^+ \otimes 560a^+, 13a^+ \otimes 428a^+)$ with $13a^+$ and $560a^+$ as above.

The relation

$$(\underline{\lambda}_1 \otimes \lambda[8,5,1])|_{b_5} = \underline{\lambda}_1 + 4\underline{\lambda}_4 + 2\lambda_{5a}$$

shows that V has a composition factor X that corresponds to $\underline{\lambda}_5$. Let S_i be an irreducible module over F affording $\underline{\lambda}_i$ for $i \in \{1,2,3,4\}$. Set $e := e_K$ and let $H := eF\tilde{A}_{14}e$ be the corresponding Hecke algebra. The dimensions $\dim S_i e = \langle \lambda_i \downarrow_K, 1_K \rangle$ of the H-modules $S_i e$ are in Table 7.6.

Table 7.6: Dimensions of $eF\tilde{A}_{14}e$-modules

i	1	2	3	4
$\dim S_i$	64	2016	4576	13664
$\dim S_i e$	4	48	88	272

We deduce from $\langle \lambda_{5a} \downarrow_K, 1_K \rangle = 1136$ that $\dim Xe \neq 0$, thus e is *faithful* for b_5. Put $a := t_{(13,1)^+} t_1 t^{-1}_{(13,1)^+} t_1$, $b := t_1 t^{-1}_{(13,1)^+}$, and $c := (ab)^3 b$. We get a C-module Ve of dimension 4636 where C is the condensation algebra generated by eae and ece. This module has the C-constituents 4a, 88a, 272a, and 1040a with multiplicities 5, 2, 4, and 2, respectively. It follows that 1040a is a composition factor of $Xe{\downarrow_C}$. Let ve be the first vector of the peakword kernel corresponding to 1040a determined by the C–MeatAxe program pwkond, and denote by v the embedding of ve into V. Spinning up v under the action of $F\tilde{A}_{14}$ yields a basis $\mathcal{B}$ of an $F\tilde{A}_{14}$-submodule $Y \leqslant V$ of dimension 50624. By (3.7), Y involves X, hence Ye involves the non-zero Xe. Calculating $\mathcal{B}e$ shows that Ye has dimension 1040 as a H-module. Thus $X = Y$ and $Xe = 1040a$ is a *genuine* H-module. The Brauer character $\underline{\lambda}_5 := \lambda_{5a} - 2\underline{\lambda}_1 - \underline{\lambda}_3$ afforded by X completes $\mathrm{IBr}(b_5)$. The decomposition matrix of b_5 is Table 7.16.

7.4 The block B_7

Since B_7 covers b_5, the degrees of the five pairs of associate irreducible Brauer characters of B_7 are 64, 2016, 4576, 13664, and 50624 (see (1.6) and (5.5.1)). Let the characters σ_i and Σ_i be defined by Table 7.7, in which $\underline{\lambda}_4(b_5)$ refers to $\underline{\lambda}_4 \in \mathrm{IBr}(b_5)$ and $\underline{\Lambda}_i(b_5)$ refers to $\underline{\Lambda}_i \in \mathrm{IPr}(b_5)$ for $i \in \{1,3,5\}$. Then $\{\sigma_1, \ldots, \sigma_{10}\}$ is a basic set of Brauer characters and $\{\Sigma_1, \ldots, \Sigma_{10}\}$ is a basic set of projective characters. Their matrix of mutual scalar products is Table 7.8.

Table 7.7: Some characters for B_7

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(14)$	64	$\underline{\Lambda}_1(b_5)\uparrow^{\tilde{S}_{14}}$	3732480
2	$\sigma(14)^a$	64	$(\sigma(14) \otimes \Sigma\langle 4^2, 3, 1^3\rangle)\|_{B_7}$	12954816
3	$\sigma[\![3^3, 2, 1 : 2, -]\!]\uparrow^{\tilde{S}_{14}}\|_{B_7}$	2144	$\Sigma[\![5, 3^2, 2, +]\!]\uparrow^{\tilde{S}_{14}}$	653184
4	$\sigma[\![3^3, 2, 1 : 2, +]\!]\uparrow^{\tilde{S}_{14}}\|_{B_7}$	2144	$\Sigma[\![5, 3^2, 2, -]\!]\uparrow^{\tilde{S}_{14}}$	653184
5	$\sigma\langle\!\langle 11, 2, 1, +\rangle\!\rangle\|_{3'}$	6720	$\Sigma[\![5, 3^2, 1 : 2, -]\!]\uparrow^{\tilde{S}_{14}}\|_{B_7}$	4144608
6	$\sigma\langle\!\langle 11, 2, 1, -\rangle\!\rangle\|_{3'}$	6720	$\underline{\Lambda}_3(b_5)\uparrow^{\tilde{S}_{14}}$	1819584
7	$\underline{\lambda}_4(b_5)\uparrow^{\tilde{S}_{14}}$	27328	$\Sigma[\![6, 5, 2, +]\!]\uparrow^{\tilde{S}_{14}}$	762048
8	$\sigma[\![3^2, 2, 1, + : 3, 2]\!]\uparrow^{\tilde{S}_{14}}\|_{B_7}$	35840	$\Sigma[\![6, 5, 2, -]\!]\uparrow^{\tilde{S}_{14}}$	762048
9	$\sigma\langle\!\langle 8, 5, 1, +\rangle\!\rangle\|_{3'}$	68992	$\Sigma[\![6, 4, 2 : 2, +]\!]\uparrow^{\tilde{S}_{14}}\|_{B_7}$	3569184
10	$\sigma\langle\!\langle 8, 5, 1, -\rangle\!\rangle\|_{3'}$	68992	$\underline{\Lambda}_5(b_5)\uparrow^{\tilde{S}_{14}}$	513216

Table 7.8: Mutual scalar products (B_7)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}
σ_1	1	.	.	.	.	.	.	.	.	.
σ_2	1	1	.	.	.	.	.	.	.	.
σ_3	2	3	1	.	.	.	.	.	.	.
σ_4	2	3	.	1	1	.	.	.	.	.
σ_5	2	4	1	.	2	1	.	.	.	.
σ_6	2	4	.	1	2	1	.	.	.	.
σ_7	.	2	.	.	1	.	1	1	3	.
σ_8	6	15	2	2	7	3	1	.	1	.
σ_9	2	13	.	.	2	1	.	1	4	1
σ_{10}	2	13	.	.	2	1	1	.	4	1

We know by (1.2) that eight irreducible Brauer characters of B_7 are invariant under complex conjugation. Thus there is only one proper pair of dual characters, namely the basic spin characters $\underline{\sigma}_1 := \sigma_1$ and $\underline{\sigma}_2 := \underline{\sigma}_1^{\circledast} = \sigma_2$. Considering the degrees of the elements of $\mathrm{IBr}(B_7)$, it is clear that an irreducible Brauer character of degree 2016 is contained in σ_3. More precisely,

$$\sigma(x, y) := \sigma_3 - x\underline{\sigma}_1 - y\underline{\sigma}_2$$

is an irreducible and necessarily self-dual Brauer character for suitable $x, y \in \{0, 1, 2\}$ with $x + y = 2$. As σ_3 is self-dual, it follows from

$$\sigma(x, y) = \sigma(x, y)^{\circledast} = \sigma_3 - x\underline{\sigma}_2 - y\underline{\sigma}_1$$

that $x = y = 1$. Hence $\underline{\sigma}_3 := \sigma_3 - \sigma(1, 1)$, and $\underline{\sigma}_4 := \underline{\sigma}_3^{a} = \sigma_4 - \sigma(1, 1)$ are irreducible. Furthermore, $\langle \sigma_5, \Sigma_1 \rangle = 2$ shows that $\underline{\sigma}_1$ and its dual $\underline{\sigma}_2$ could be contained in σ_5, in total at most two times, and $\underline{\sigma}_4$ cannot be contained in σ_5 as this would lead to $\langle \sigma_5 - \underline{\sigma}_4, \Sigma_4 \rangle = -1$. Therefore the irreducible constituent of σ_5 of degree 4576 is

$$\sigma(x, y) := \sigma_5 - x\underline{\sigma}_1 - y\underline{\sigma}_2 - \underline{\sigma}_3$$

for suitable $x, y \in \{0, 1, 2\}$ with $x + y = 2$. Again, the self-duality of $\underline{\sigma}_3$, σ_5 and $\sigma(x, y)$ forces $x = y = 1$. Thus we get the irreducibles $\underline{\sigma}_5 := \sigma(1, 1)$, $\underline{\sigma}_6 := \underline{\sigma}_5^{a} = \sigma_6 - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_4$. The relations

$$\underline{\Lambda}_2(b_5)\uparrow^{\tilde{S}_{14}} = \Sigma_3 + \Sigma_4,$$
$$\underline{\Lambda}_4(b_5)\uparrow^{\tilde{S}_{14}} = \Sigma_7 + \Sigma_8$$

show that $\underline{\Sigma}_3 := \Sigma_3$, $\underline{\Sigma}_4 := \underline{\Sigma}_3^{a} = \Sigma_4$, $\underline{\Sigma}_7 := \Sigma_7$, and $\underline{\Sigma}_8 := \underline{\Sigma}_7^{a} = \Sigma_8$ are indecomposable, see (1.7). Moreover, as $\Sigma_{10} = \underline{\Lambda}_5(b_5)\uparrow^{\tilde{S}_{14}}$ is the sum of two projective atoms $\underline{\Sigma}_9 := \sigma_9^*$ and $\underline{\Sigma}_{10} := \underline{\Sigma}_9^{a} = \sigma_{10}^*$. These are projective indecomposable characters by (2.8). Table 7.9 contains the mutual scalar products of the basic sets $\{\underline{\sigma}_1, \underline{\sigma}_2, \underline{\sigma}_3, \underline{\sigma}_4, \underline{\sigma}_5, \underline{\sigma}_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$ and $\{\Sigma_1, \Sigma_2, \underline{\Sigma}_3, \underline{\Sigma}_4, \Sigma_5, \Sigma_6, \underline{\Sigma}_7, \underline{\Sigma}_8, \underline{\Sigma}_9, \underline{\Sigma}_{10}\}$.

As $\Sigma_1\downarrow_{\tilde{A}_{14}} = 2\underline{\Lambda}_1(b_5)$, we know that Σ_1 is the sum of the two projective indecomposable characters of degree $\deg \underline{\Lambda}_1(b_5) = 1866240$ which extend $\underline{\Lambda}_1(b_5)$ to $\tilde{S}_{14}$, see (1.7). Σ_1 has

Table 7.9: Mutual scalar products (B_7)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	Σ_5	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	1	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	2	1	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	2	.	1	1	.	.	.	.	.
$\underline{\sigma}_5$	.	1	.	.	2	1	.	.	.	.
$\underline{\sigma}_6$	.	1	.	.	1	1	.	.	.	.
σ_7	.	2	.	.	1	.	1	1	.	.
σ_8	6	15	2	2	7	3	1	.	.	.
σ_9	2	13	.	.	2	1	.	1	1	.
σ_{10}	2	13	.	.	2	1	1	.	.	1

42 distinct parts of degree 1866240, each of the form

$$\Sigma(x_1, x_2, x_8, x_9, x_{10}) := x_1\underline{\sigma}_1^* + x_2\underline{\sigma}_2^* + x_8\sigma_8^* + x_9\sigma_9^* + x_{10}\sigma_{10}^*$$

with integers $0 \leqslant x_1, x_2 \leqslant 1$, $0 \leqslant x_8 \leqslant 6$, and $0 \leqslant x_9, x_{10} \leqslant 2$. Each of these parts is non-self-dual and non-self-associate, but there is only one pair $\{\Phi, \Psi\}$ such that $\Sigma_1 = \Phi + \Psi$ and $\Phi^a = \Phi^{\circledast} = \Psi$. These are $\underline{\Sigma}_1 := \Sigma(0,1,3,1,1)$ and $\underline{\Sigma}_2 := \underline{\Sigma}_1^a = \Sigma(1,0,3,1,1)$. We exchange $\{\Sigma_1, \Sigma_2\}$ with $\{\underline{\Sigma}_1, \underline{\Sigma}_2\}$ and get Table 7.10.

Table 7.10: Mutual scalar products (B_7)

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	Σ_5	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	.	1	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	.	1	1	.	.	.	.	.
$\underline{\sigma}_5$	.	.	.	.	2	1	.	.	.	.
$\underline{\sigma}_6$	.	.	.	.	1	1	.	.	.	.
σ_7	.	.	.	.	1	.	1	1	.	.
σ_8	3	3	2	2	7	3	1	.	.	.
σ_9	1	1	.	.	2	1	.	1	1	.
σ_{10}	1	1	.	.	2	1	1	.	.	1

Now we can use the known pairs $(\underline{\sigma}_i, \underline{\Sigma}_i)$ such that $\langle \underline{\sigma}_i, \underline{\Sigma}_i \rangle = 1$ for $i \in \{1, \dots, 4\}$ to improve the basic sets. Clearly,

$$\sigma_{8a} := \sigma_8 - 3\underline{\sigma}_1 - 3\underline{\sigma}_2 - 2\underline{\sigma}_3 - 2\underline{\sigma}_4,$$
$$\sigma_{9a} := \sigma_9 - \underline{\sigma}_1 - \underline{\sigma}_2,$$
$$\sigma_{10a} := \sigma_{10} - \underline{\sigma}_1 - \underline{\sigma}_2$$

are Brauer characters and $\Sigma_{5a} := \Sigma_5 - \underline{\Sigma}_4$ is projective. We exchange $\{\sigma_8, \sigma_9, \sigma_{10}\}$ with $\{\sigma_{8a}, \sigma_{9a}, \sigma_{10a}\}$ and Σ_5 with Σ_{5a} and proceed by looking at Table 7.11.

Table 7.11: Mutual scalar products (B_7)

$\langle \cdot, \cdot \rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	Σ_{5a}	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_5$	.	.	.	.	2	1	.	.	.	.
$\underline{\sigma}_6$	.	.	.	.	1	1	.	.	.	.
σ_7	.	.	.	.	1	.	1	1	.	.
σ_{8a}	.	.	.	.	5	3	1	.	.	.
σ_{9a}	.	.	.	.	2	1	.	1	1	.
σ_{10a}	.	.	.	.	2	1	1	.	.	1

As $\underline{\lambda}_4(b_5)\uparrow^{\tilde{S}_{14}} = \sigma_7$, a pair of irreducible Brauer characters $\underline{\sigma}_7$ and $\underline{\sigma}_7^a$ is contained in σ_7. We have $\sigma_7 = \Sigma_{5a}^* + \underline{\Sigma}_7^* + \underline{\Sigma}_8^*$ with $\deg \Sigma_{5a}^* = 0$ and $\deg \underline{\Sigma}_7^* = \deg \underline{\Sigma}_8^* = 13664$. Without loss, either $\underline{\sigma}_7 = \Sigma_{5a}^* + \underline{\Sigma}_8^*$ and $\underline{\sigma}_7^a = \underline{\Sigma}_7^*$, or $\underline{\sigma}_7 = \Sigma_{5a}^* + \underline{\Sigma}_7^*$ and $\underline{\sigma}_7^a = \underline{\Sigma}_8^*$. Both possibilities can be distinguished by the Brauer character value on a class of type $(7, 5, 2)$. There only the latter pair has zero values. We find an irreducible 13664-dimensional representation of $\tilde{S}_{14}$ as a constituent of $64a^- \otimes 428a^+$ over $GF(9)$, where $428a^+$ is a constituent of the tensor product of $13a^+$ and $560a^+$ (for their construction see 7.2). This shows that $\underline{\sigma}_7 := \Sigma_{5a}^* + \underline{\Sigma}_7^*$ and $\underline{\sigma}_8 := \underline{\sigma}_7^a = \underline{\Sigma}_8^*$ are irreducible Brauer characters. We exchange $\{\sigma_7, \sigma_{8a}\}$ with $\{\underline{\sigma}_7, \underline{\sigma}_8\}$. The mutual scalar products (Table 7.12) show that we may use the pairs $(\underline{\sigma}_7, \underline{\Sigma}_7)$ and $(\underline{\sigma}_8, \underline{\Sigma}_8)$ for further improvements.
We get $\sigma_{9b} := \sigma_{9a} - \underline{\sigma}_8$ and $\sigma_{10b} := \sigma_{10a} - \underline{\sigma}_7$ instead of σ_{9a} and σ_{10a}, and $\Sigma_{5b} := \Sigma_{5a} - \underline{\Sigma}_7$ instead of Σ_{5a}. Finally, the relation

$$\underline{\lambda}_5(b_5)\uparrow^{\tilde{S}_{14}} = -\underline{\sigma}_5 - \underline{\sigma}_6 + \sigma_{9b} + \sigma_{10b}$$

Table 7.12: Mutual scalar products (B_7)

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	Σ_{5a}	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_5$	.	.	.	.	2	1	.	.	.	.
$\underline{\sigma}_6$	.	.	.	.	1	1	.	.	.	.
$\underline{\sigma}_7$	.	.	.	.	1	.	1	.	.	.
$\underline{\sigma}_8$	.	.	.	.	.	.	.	1	.	.
σ_{9a}	.	.	.	.	2	1	.	1	1	.
σ_{10a}	.	.	.	.	2	1	1	.	.	1

and the mutual scalar products in Table 7.13 imply that $\underline{\sigma}_5$ is contained in σ_{9b} and $\underline{\sigma}_6$ is contained in σ_{10b}. Thus $\underline{\sigma}_9 := \sigma_{9b} - \underline{\sigma}_5$ and $\underline{\sigma}_{10} := \underline{\sigma}_9^a = \sigma_{10b} - \underline{\sigma}_6$ are irreducible. The decomposition matrix of B_7 is Table 7.17.

Table 7.13: Mutual scalar products (B_7)

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	Σ_{5b}	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_5$	.	.	.	.	2	1	.	.	.	.
$\underline{\sigma}_6$	.	.	.	.	1	1	.	.	.	.
σ_{9b}	.	.	.	.	2	1	.	.	1	.
σ_{10b}	.	.	.	.	1	1	.	.	.	1

Table 7.14: 3-modular decomposition matrix of block B_8 of $\tilde{S}_{14}$ (defect 4, bar core $(4,1)$, weight 3, content $(10,4)$)

			$[\![4,3^3,1]\!]$ 768	$[\![6,4,3,1]\!]$ 26688	$[\![7,4,2,1]\!]$ 30912
$\star$	768	$\langle\!\langle 13,1\rangle\!\rangle$	1	.	.
$\star$	27456	$\langle\!\langle 10,4\rangle\!\rangle$	1	1	.
	28224	$\langle\!\langle 10,3,1,+\rangle\!\rangle$	2	1	.
	28224	$\langle\!\langle 10,3,1,-\rangle\!\rangle$	2	1	.
	59136	$\langle\!\langle 9,4,1,+\rangle\!\rangle$	2	1	1
	59136	$\langle\!\langle 9,4,1,-\rangle\!\rangle$	2	1	1
$\star$	31680	$\langle\!\langle 7,6,1,+\rangle\!\rangle$	1	.	1
	31680	$\langle\!\langle 7,6,1,-\rangle\!\rangle$	1	.	1
	59904	$\langle\!\langle 7,4,3,+\rangle\!\rangle$	3	1	1
	59904	$\langle\!\langle 7,4,3,-\rangle\!\rangle$	3	1	1
	87360	$\langle\!\langle 7,4,2,1\rangle\!\rangle$	4	2	1
	54912	$\langle\!\langle 6,4,3,1\rangle\!\rangle$	2	2	.

Table 7.15: 3-modular decomposition matrix of block $\mathfrak{b}_6$ of $\tilde{A}_{14}$ (defect 4, bar core $(4,1)$, weight 3, content $(10,4)$)

			$[\![4,3^3,1,+]\!]$ 384	$[\![4,3^3,1,-]\!]$ 384	$[\![6,4,3,1,+]\!]$ 13344	$[\![6,4,3,1,-]\!]$ 13344	$[\![7,4,2,1,+]\!]$ 15456	$[\![7,4,2,1,-]\!]$ 15456
$\star$	384	$\langle\!\langle 13,1,+\rangle\!\rangle$	.	1	.	.	.	.
$\star$	384	$\langle\!\langle 13,1,-\rangle\!\rangle$	1	.	.	.	.	.
$\star$	13728	$\langle\!\langle 10,4,+\rangle\!\rangle$	.	1	1	.	.	.
$\star$	13728	$\langle\!\langle 10,4,-\rangle\!\rangle$	1	.	.	1	.	.
	28224	$\langle\!\langle 10,3,1\rangle\!\rangle$	2	2	1	1	.	.
	59136	$\langle\!\langle 9,4,1\rangle\!\rangle$	2	2	1	1	1	1
$\star$	31680	$\langle\!\langle 7,6,1\rangle\!\rangle$	1	1	.	.	1	1
	59904	$\langle\!\langle 7,4,3\rangle\!\rangle$	3	3	1	1	1	1
	43680	$\langle\!\langle 7,4,2,1,+\rangle\!\rangle$	2	2	1	1	1	.
$\star$	43680	$\langle\!\langle 7,4,2,1,-\rangle\!\rangle$	2	2	1	1	.	1
	27456	$\langle\!\langle 6,4,3,1,+\rangle\!\rangle$	1	1	1	1	.	.
	27456	$\langle\!\langle 6,4,3,1,-\rangle\!\rangle$	1	1	1	1	.	.

Table 7.16: 3-modular decomposition matrix of block $\mathfrak{b}_5$ of $\tilde{A}_{14}$
(defect 5, bar core (2), weight 4, content (9, 5))

			$[\![3^4, 2]\!]$ 64	$[\![5, 3^2, 2, 1]\!]$ 2016	$[\![5, 4, 3, 2]\!]$ 4576	$[\![6, 5, 2, 1]\!]$ 13664	$[\![7, 5, 2]\!]$ 50624
$\star$	64	$\langle\!\langle 14 \rangle\!\rangle$	1	.	.	.	.
$\star$	2080	$\langle\!\langle 12, 2, + \rangle\!\rangle$	1	1	.	.	.
	2080	$\langle\!\langle 12, 2, - \rangle\!\rangle$	1	1	.	.	.
	6656	$\langle\!\langle 11, 3, + \rangle\!\rangle$	1	1	1	.	.
	6656	$\langle\!\langle 11, 3, - \rangle\!\rangle$	1	1	1	.	.
	6720	$\langle\!\langle 11, 2, 1 \rangle\!\rangle$	2	1	1	.	.
	18304	$\langle\!\langle 9, 5, + \rangle\!\rangle$	1	.	1	1	.
	18304	$\langle\!\langle 9, 5, - \rangle\!\rangle$	1	.	1	1	.
	40768	$\langle\!\langle 9, 3, 2 \rangle\!\rangle$	4	2	2	2	.
$\star$	13728	$\langle\!\langle 8, 6, + \rangle\!\rangle$	1	.	.	1	.
	13728	$\langle\!\langle 8, 6, - \rangle\!\rangle$	1	.	.	1	.
$\star$	68992	$\langle\!\langle 8, 5, 1 \rangle\!\rangle$	2	.	1	1	1
	96096	$\langle\!\langle 8, 4, 2 \rangle\!\rangle$	6	2	3	2	1
	20384	$\langle\!\langle 8, 3, 2, 1, + \rangle\!\rangle$	2	1	1	1	.
	20384	$\langle\!\langle 8, 3, 2, 1, - \rangle\!\rangle$	2	1	1	1	.
	91520	$\langle\!\langle 7, 5, 2 \rangle\!\rangle$	6	2	2	2	1
	40768	$\langle\!\langle 6, 5, 3 \rangle\!\rangle$	4	2	2	2	.
	24960	$\langle\!\langle 6, 5, 2, 1, + \rangle\!\rangle$	2	1	2	1	.
	24960	$\langle\!\langle 6, 5, 2, 1, - \rangle\!\rangle$	2	1	2	1	.
$\star$	4576	$\langle\!\langle 5, 4, 3, 2, + \rangle\!\rangle$	.	.	1	.	.
	4576	$\langle\!\langle 5, 4, 3, 2, - \rangle\!\rangle$	.	.	1	.	.

Table 7.17: 3-modular decomposition matrix of block B_7 of $\tilde{S}_{14}$ (defect 5, bar core (2), weight 4, content $(9,5)$)

			$[\![3^4,2,+]\!]$ 64	$[\![3^4,2,-]\!]$ 64	$[\![5,3^2,2,1,+]\!]$ 2016	$[\![5,3^2,2,1,-]\!]$ 2016	$[\![5,4,3,2,+]\!]$ 4576	$[\![5,4,3,2,-]\!]$ 4576	$[\![6,5,2,1,+]\!]$ 13664	$[\![6,5,2,1,-]\!]$ 13664	$[\![7,5,2,+]\!]$ 50624	$[\![7,5,2,-]\!]$ 50624
$\star$	64	$\langle\!\langle 14,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
$\star$	64	$\langle\!\langle 14,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.
$\star$	4160	$\langle\!\langle 12,2\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.
	13312	$\langle\!\langle 11,3\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.
$\star$	6720	$\langle\!\langle 11,2,1,+\rangle\!\rangle$	1	1	.	1	1	.	.	.	.	.
$\star$	6720	$\langle\!\langle 11,2,1,-\rangle\!\rangle$	1	1	1	.	.	1	.	.	.	.
	36608	$\langle\!\langle 9,5\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.
	40768	$\langle\!\langle 9,3,2,+\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	.
	40768	$\langle\!\langle 9,3,2,-\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	.
$\star$	27456	$\langle\!\langle 8,6\rangle\!\rangle$	1	1	.	.	.	.	1	1	.	.
$\star$	68992	$\langle\!\langle 8,5,1,+\rangle\!\rangle$	1	1	.	.	1	.	.	1	.	1
$\star$	68992	$\langle\!\langle 8,5,1,-\rangle\!\rangle$	1	1	.	.	.	1	1	.	1	.
$\star$	96096	$\langle\!\langle 8,4,2,+\rangle\!\rangle$	3	3	1	1	1	2	1	1	1	.
	96096	$\langle\!\langle 8,4,2,-\rangle\!\rangle$	3	3	1	1	2	1	1	1	.	1
	40768	$\langle\!\langle 8,3,2,1\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	.
	91520	$\langle\!\langle 7,5,2,+\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	.
$\star$	91520	$\langle\!\langle 7,5,2,-\rangle\!\rangle$	3	3	1	1	1	1	1	1	.	1
	40768	$\langle\!\langle 6,5,3,+\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	.
	40768	$\langle\!\langle 6,5,3,-\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	.
	49920	$\langle\!\langle 6,5,2,1\rangle\!\rangle$	2	2	1	1	2	2	1	1	.	.
	9152	$\langle\!\langle 5,4,3,2\rangle\!\rangle$	.	.	.	.	1	1	.	.	.	.

8 The 5-modular spin characters of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

Table 8.1 contains the block constants of the spin 5-blocks of $\tilde{S}_{14}$ and $\tilde{A}_{14}$.

Table 8.1: Spin 5-blocks of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
21	13	10	2	(4)	2	12	11	5
22	11	5	2	$(3,1)$	2	13	13	10
23	5	4	1	$(6,2,1)$	1	14	4	2
24	4	2	1	$(7,2)$	1	15	5	4

The 5-modular basic spin characters of $\tilde{S}_{14}$ are $\sigma(14) := \sigma\langle\!\langle 14,+\rangle\!\rangle|_{5'}$, $\sigma(14)^a = \sigma\langle\!\langle 14,-\rangle\!\rangle|_{5'}$. They restrict irreducibly to $\tilde{A}_{14}$ as $\lambda(14) := \lambda\langle\!\langle 14\rangle\!\rangle|_{5'}$. Also, $\sigma(13,1) := \sigma\langle\!\langle 13,1\rangle\!\rangle|_{5'}$ is irreducible, and $\sigma\langle\!\langle 13,1\rangle\!\rangle\downarrow_{\tilde{A}_{14}} = \lambda\langle\!\langle 13,1,+\rangle\!\rangle + \lambda\langle\!\langle 13,1,-\rangle\!\rangle$ yields the irreducible Brauer characters $\lambda(13,1) := \lambda\langle\!\langle 13,1,+\rangle\!\rangle|_{5'}$ and $\lambda\langle\!\langle 13,1,-\rangle\!\rangle|_{5'}$ of $\tilde{A}_{14}$.

8.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{23}, B_{24}, b_{14} and b_{15} are given in Appendix A as Figures A.1, A.2, A.3 and A.4, respectively.

8.2 The block B_{22}

Table 8.2 defines Brauer characters $\underline{\sigma}_i$ and projective characters $\underline{\Sigma}_i$ for B_{22}. We have $\langle \underline{\sigma}_i, \underline{\Sigma}_j\rangle = \delta_{ij}$ for all $i, j \in \{1,\ldots,5\}$. Thus $\underline{\sigma}_i$ is irreducible and $\underline{\Sigma}_i$ is indecomposable for $i \in \{1,\ldots,5\}$. The decomposition matrix of B_{22} is Table 8.9.

8.3 The block b_{13}

The block b_{13} is covered by B_{22}. Thus we know that there are five pairs of conjugate irreducible Brauer characters in b_{13}, and their degrees are 384, 6272, 7456, 7840, and 12544.

Table 8.2: Some characters for B_{22}

i	$\underline{\sigma}_i$	$\deg \underline{\sigma}_i$	$\underline{\Sigma}_i$	$\deg \underline{\Sigma}_i$
1	$\sigma(13,1)$	768	$\sigma(14) \otimes \Sigma[12,2] \mid_{B_{22}}$	233600
2	$\sigma[\![7,5,1,-]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	12544	$\Sigma[\![8,3,+:3]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	235200
3	$\sigma(14) \otimes \sigma[5^2,4] \mid_{B_{22}}$	14912	$\Sigma[\![5^2,2,1,-]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	356800
4	$\sigma[\![7,3,1:2,1,+]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	15680	$\Sigma[\![7,3,1:2,1,+]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	235200
5	$\sigma[\![6,4,2,1,+]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	25088	$\Sigma[\![6,4,2,1,+]\!]\uparrow^{\tilde{S}_{14}} \mid_{B_{22}}$	315200

We start with a basic set $\{\lambda_1, \dots \lambda_{10}\}$ of Brauer characters and a basic set $\{\Lambda_1, \dots \Lambda_{10}\}$ of projective characters defined by Table 8.3.

Table 8.3: Some characters for b_{13}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(13,1)$	384	$\Lambda[\![5^2,1,+:3,-]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	763200
2	$\lambda(13,1)^c$	384	$\lambda(14) \otimes \Lambda[11,2,1] \mid_{b_{13}}$	233600
3	$\lambda\langle\!\langle 11,3,+\rangle\!\rangle\vert_{5'}$	6656	$\Lambda[\![5^2,3,+]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	412800
4	$\lambda\langle\!\langle 11,3,-\rangle\!\rangle\vert_{5'}$	6656	$\Lambda[\![7,5,1]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	235200
5	$\lambda[\![8,3,2+]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	7840	$\Lambda[\![8,3,2,+]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	117600
6	$\lambda[\![8,3,2-]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	7840	$\Lambda[\![8,3,2,-]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	117600
7	$\lambda[\![6,4,3+]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	12544	$\Lambda[\![6,4,3,+]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	157600
8	$\lambda[\![6,4,3,-]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	12544	$\Lambda[\![6,4,3,-]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	157600
9	$\lambda\langle\!\langle 8,6,+\rangle\!\rangle\vert_{5'}$	13728	$\Lambda[\![5,4,3,+:2]\!]\uparrow^{\tilde{A}_{14}} \mid_{b_{13}}$	356800
10	$\lambda\langle\!\langle 8,6,-\rangle\!\rangle\vert_{5'}$	13728	$\lambda\langle 4,2^3,1^4\rangle \otimes \lambda\langle\!\langle 8,6,+\rangle\!\rangle\vert_{5'} \mid_{b_{13}}$	43508000
11			$\lambda(14) \otimes \Lambda[6,4,2,1^2] \mid_{b_{13}}$	1081600

Note that $(4,2^3,1^4) \in \mathcal{P}(14)$ is a 5-core, thus Λ_{10} is projective. The mutual scalar products of the basic set characters are given in Table 8.4.

We know that $\underline{\lambda}_1 := \lambda_1$ and its conjugate character $\underline{\lambda}_2 := \lambda_2$ are irreducible. Moreover, $\underline{\sigma}_4(B_{22}) \downarrow_{\tilde{A}_{14}} = \lambda_5 + \lambda_6$ and $\underline{\sigma}_5(B_{22}) \downarrow_{\tilde{A}_{14}} = \lambda_7 + \lambda_8$. Thus $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{5,6,7,8\}$. Then $\underline{\Lambda}_i := \Lambda_i$ is indecomposable with $\langle \underline{\lambda}_i, \underline{\Lambda}_i \rangle = 1$ for $i \in \{5,6,7,8\}$. We

Table 8.4: Mutual scalar products (b_{13})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
λ_1	2	1	.	.	.	.	.	.	.	4
λ_2	2	1	1	.	.	.	.	.	.	4
λ_3	2	1	1	1	.	.	.	.	.	44
λ_4	3	1	1	1	.	.	.	.	.	44
λ_5	.	.	.	.	1	.	.	.	.	38
λ_6	.	.	.	.	.	1	.	.	.	38
λ_7	.	.	.	.	.	.	1	.	.	48
λ_8	.	.	.	.	.	.	.	1	.	48
λ_9	1	.	1	1	.	.	.	.	1	66
λ_{10}	1	.	1	1	.	.	.	.	1	65

use these pairs $(\underline{\lambda}_i, \underline{\Lambda}_i)$ to improve Λ_{11} by

$$\Lambda_{11a} := \Lambda_{11} - \sum_{i=5}^{8} \langle \underline{\lambda}_i, \Lambda_{11} \rangle \underline{\Lambda}_i = \Lambda_{11} - 2\underline{\Lambda}_5 - 2\underline{\Lambda}_6 - \underline{\Lambda}_7 - \underline{\Lambda}_8.$$

As $\underline{\Sigma}_3(B_{22}) \downarrow_{\tilde{A}_{14}} = \Lambda_9$ is the sum of two projective atoms, the atoms $\underline{\Lambda}_9 := \lambda_9^*$ and $\underline{\Lambda}_{10} := \lambda_{10}^*$ are projective indecomposable characters. Exchanging $\{\Lambda_1, \Lambda_9, \Lambda_{10}\}$ with $\{\underline{\Lambda}_9, \underline{\Lambda}_{10}, \Lambda_{11a}\}$ yields the mutual scalar products in Table 8.5.

Table 8.5: Mutual scalar products (b_{13})

$\langle\cdot,\cdot\rangle$	Λ_2	Λ_3	Λ_4	Λ_{11a}	$\underline{\Lambda}_5$	$\underline{\Lambda}_6$	$\underline{\Lambda}_7$	$\underline{\Lambda}_8$	$\underline{\Lambda}_9$	$\underline{\Lambda}_{10}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	1	1	.	.	.	.	.	.	.	.
λ_3	1	1	1	.	.	.	.	.	.	.
λ_4	1	1	1	1	.	.	.	.	.	.
$\underline{\lambda}_5$	.	.	.	.	1	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	.	1	.	.	.	.
$\underline{\lambda}_7$	.	.	.	.	.	.	1	.	.	.
$\underline{\lambda}_8$	.	.	.	.	.	.	.	1	.	.
λ_9	.	1	1	1	.	.	.	.	1	.
λ_{10}	.	1	1	1	.	.	.	.	.	1

The equation

$$\lambda_3\uparrow^{\tilde{S}_{14}} = \underline{\sigma}_1(B_{22}) + \underline{\sigma}_2(B_{22}) = \lambda_4\uparrow^{\tilde{S}_{14}}$$

shows that the pair of irreducible Brauer characters $\{\underline{\lambda}_3, \underline{\lambda}_4\}$ of degree 6272 is either $\{\lambda_3 - \underline{\lambda}_1, \lambda_4 - \underline{\lambda}_2\}$ or $\{\lambda_3 - \underline{\lambda}_2, \lambda_4 - \underline{\lambda}_1\}$. Likewise,

$$\lambda_9\uparrow^{\tilde{S}_{14}} = \underline{\sigma}_2(B_{22}) + \underline{\sigma}_3(B_{22}) = \lambda_{10}\uparrow^{\tilde{S}_{14}},$$

shows that the irreducible Brauer characters of degree 7456 are $\{\lambda_9 - \underline{\lambda}_3, \lambda_{10} - \underline{\lambda}_4\}$ or $\{\lambda_9 - \underline{\lambda}_4, \lambda_{10} - \underline{\lambda}_3\}$. This gives four different possibilities for the latter pair. The candidates can be distinguished by their character values on classes of type $(8,6)$, $(13,1)^+$ and $(13,1)^-$ and also by their traces. Let $13a^+$ be the reduced permutation module of S_{14}. We find a 233-dimensional irreducible module $233a^+$ of $\tilde{S}_{14}$ over $GF(5)$ that affords $\sigma[7^2]$ by chopping the sequence $(13a^+ \otimes 13a^+, 13a^+ \otimes 77a^+, 13a^+ \otimes 196a^+, 13a^+ \otimes 624a^+, 13a^+ \otimes 1001a^+, 13a^+ \otimes 377a^+)$. The tensor product of $233a^+$ with a basic spin module affords $\underline{\sigma}_3$, and its restriction to $\tilde{A}_{14}$ is irreducible over $GF(5)$ but has two constituents of dimension 7456 over $GF(25)$. We compare the traces of $t_{(8,6)}$, $t_{(11,3)^+}$ and $t_{(11,3)^-}$ on one of them with the character values on the corresponding classes. This identifies $\underline{\lambda}_9 := \lambda_9 - \lambda_4 + \underline{\lambda}_1$ as the corresponding irreducible Brauer character and $\underline{\lambda}_{10} := \lambda_{10} - \lambda_3 + \underline{\lambda}_2$. In particular, $\underline{\lambda}_3 := \lambda_9 - \underline{\lambda}_9 = \lambda_4 - \underline{\lambda}_1$, and $\underline{\lambda}_4 := \lambda_3 - \underline{\lambda}_2$ complete $\mathrm{IBr}(b_{13})$. The decomposition matrix of b_{13} is Table 8.10.

8.4 The block b_{12}

Table 8.6 defines Brauer characters $\underline{\lambda}_i$ and projective characters $\underline{\Lambda}_i$ for $i \in \{1, \ldots, 5\}$. Note that $\lambda\langle\langle 9,4,2,+\rangle\rangle$ and $\lambda\langle\langle 8,4,3,+\rangle\rangle$ are ordinary $\tilde{A}_{15}$-characters of defect 0. As $\langle \underline{\lambda}_i, \underline{\Lambda}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, 5\}$, we are done. The decomposition matrix of b_{12} is Table 8.11.

8.5 The block B_{21}

As B_{21} covers b_{12}, we need to find five pairs of associate, irreducible Brauer characters for B_{21}. Their degrees are 64, 4576, 13664, 40768, and 50624. One pair of irreducibles are $\underline{\sigma}_1 := \sigma(14)$ and $\underline{\sigma}_1^a$. We define further Brauer characters σ_i and projective characters Σ_i by Table 8.7.

Table 8.6: Some characters for $\mathfrak{b}_{12}$

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg \underline{\Lambda}_i$
1	$\lambda(14)$	64	$\Lambda[\![5^2, 3, -]\!]\uparrow^{\tilde{A}_{14}} \vert_{\mathfrak{b}_{12}}$	494400
2	$\lambda\langle\!\langle 5, 4, 3, 2, -\rangle\!\rangle\vert_{5'}$	4576	$\Lambda[\![5, 4, 3, 1, +]\!]\uparrow^{\tilde{A}_{14}} \vert_{\mathfrak{b}_{12}}$	320800
3	$\lambda[\![6, 4, 1, + : 3+]\!]\uparrow^{\tilde{A}_{14}} \vert_{\mathfrak{b}_{12}}$	13664	$\Lambda[\![9, 4]\!]\uparrow^{\tilde{A}_{14}}$	123200
4	$\lambda\langle\!\langle 9, 3, 2\rangle\!\rangle\vert_{5'}$	40768	$\lambda\langle\!\langle 9, 4, 2, +\rangle\!\rangle\downarrow_{\tilde{A}_{14}}$	196000
5	$\lambda[\![6, 4, 3, -]\!]\uparrow^{\tilde{A}_{14}} \vert_{\mathfrak{b}_{12}}$	50624	$\lambda\langle\!\langle 8, 4, 3, +\rangle\!\rangle\downarrow_{\tilde{A}_{14}}$	156000

Table 8.7: Some characters for B_{21}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1			$\sigma(14) \otimes \Sigma[12, 1^2] \vert_{B_{21}}$	494400
2	$\sigma[\![4, 3, 2, 1 : 4, +]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	9152	$\Sigma[\![5, 4, 3 : 2, +]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	444000
3	$\sigma[\![5, 4, 1 : 4, +]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	13792	$\Sigma[\![9, 4, -]\!]\uparrow^{\tilde{S}_{14}}$	123200
4	$\sigma[\![5, 4 : 3, 2+]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	40768	$\Sigma[\![8, 4 : 2, -]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	565600
5	$\sigma\langle\!\langle 7, 4, 3, +\rangle\!\rangle\vert_{5'}$	59904	$\Sigma[\![7, 4, - : 3]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	721600
6	$\sigma[\![5, 4 : 4, 1, +]\!]\uparrow^{\tilde{S}_{14}} \vert_{B_{21}}$	45824		

Table 8.8: Mutual scalar products (B_{21})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_1^a	Σ_2	Σ_2^a	Σ_3^a	Σ_3	Σ_4	Σ_4^a	Σ_5	Σ_5^a
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_1^a$	.	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	.	1	.	.	.	.	.	.	.
$\underline{\sigma}_2^a$	.	.	.	1	.	.	.	.	.	.
σ_3	1	1	1	.	1	.	1	2	1	2
σ_3^a	1	1	.	1	.	1	2	1	2	1
$\underline{\sigma}_4$	.	.	.	.	.	.	1	.	1	.
$\underline{\sigma}_4^a$	.	.	.	.	.	.	.	1	.	1
σ_5^a	1	1	1	1	.	.	.	.	1	.
σ_5	1	1	1	1	.	.	.	.	.	1

Then we have the basic sets $\{\underline{\sigma}_1, \underline{\sigma}_1^a, \sigma_2, \sigma_3, \sigma_3^a, \sigma_4, \sigma_4^a, \sigma_5, \sigma_5^a, \sigma_6\}$ and $\{\Sigma_1, \Sigma_1^a, \dots, \Sigma_5, \Sigma_5^a\}$. As $\sigma_2 = \sigma_2^a = \underline{\lambda}_2(b_{12})\uparrow^{\tilde{S}_{14}} = \Sigma_2^* + (\Sigma_2^a)^*$, the Brauer atoms $\underline{\sigma}_2 := \Sigma_2^*$ and $\underline{\sigma}_2^a = (\Sigma_2^a)^*$ are irreducible Brauer characters. Moreover, $\underline{\sigma}_4 := \sigma_4$ and its associate character are irreducible since $\sigma_4\downarrow_{\tilde{A}_{14}} = \underline{\lambda}_4(b_{12})$. After exchanging $\{\sigma_2, \sigma_6\}$ with $\{\underline{\sigma}_2, \underline{\sigma}_2^a\}$, we get the mutual scalar products in Table 8.8. The decomposition $\sigma_3\downarrow_{\tilde{A}_{14}} = 2\underline{\lambda}_1 + \underline{\lambda}_3$ shows

$$\sigma_3 = x\underline{\sigma}_1 + x_a\underline{\sigma}_1^a + \underline{\sigma}_3$$

for suitable $x, x_a \in \mathbb{Z}$ with $0 \leqslant x, x_a \leqslant 2$ and $x + x_a = 2$. The decomposition of σ_3 in terms of the Brauer atoms given by the scalar products in Table 8.8 determines immediately $x = x_a = 1$. Analogously, $\sigma_5\downarrow_{\tilde{A}_{14}} = 2\underline{\lambda}_1 + 2\underline{\lambda}_2 + \underline{\lambda}_5$ implies

$$\sigma_5 = x\underline{\sigma}_1 + x_a\underline{\sigma}_1^a + y\underline{\sigma}_2 + y_a\underline{\sigma}_2^a + \underline{\sigma}_5$$

for suitable $x, x_a, y, y_a \in \mathbb{Z}$ with $0 \leqslant x, x_a, y, y_a \leqslant 2$ and $x + x_a = y + y_a = 2$. Here we can deduce that $x = x_a = y = y_a = 1$. Hence $\underline{\sigma}_3 := \sigma_3 - \underline{\sigma}_1 - \underline{\sigma}_1^a$ and $\underline{\sigma}_5 := \sigma_5 - \underline{\sigma}_1 - \underline{\sigma}_1^a - \underline{\sigma}_2 - \underline{\sigma}_2^a$, together with their associate characters, complete $\mathrm{IBr}(B_{21})$. The decomposition matrix of B_{21} is Table 8.12.

Table 8.9: 5-modular decomposition matrix of block B_{22} of $\tilde{S}_{14}$
(defect 2, bar core $(3, 1)$, weight 2, content $(6, 5, 3)$)

			$[\![5^2, 3, 1]\!]$ 14912	$[\![6, 4, 3, 1]\!]$ 25088	$[\![6, 5, 3]\!]$ 768	$[\![8, 3, 2, 1]\!]$ 15680	$[\![8, 5, 1]\!]$ 12544
$\star$	768	$\langle\!\langle 13, 1\rangle\!\rangle$	.	.	1	.	.
$\star$	13312	$\langle\!\langle 11, 3\rangle\!\rangle$	.	.	1	.	1
$\star$	28224	$\langle\!\langle 10, 3, 1, +\rangle\!\rangle$	.	.	.	1	1
	28224	$\langle\!\langle 10, 3, 1, -\rangle\!\rangle$	.	.	.	1	1
$\star$	27456	$\langle\!\langle 8, 6\rangle\!\rangle$	1	.	.	.	1
	68992	$\langle\!\langle 8, 5, 1, +\rangle\!\rangle$	1	1	1	1	1
	68992	$\langle\!\langle 8, 5, 1, -\rangle\!\rangle$	1	1	1	1	1
	40768	$\langle\!\langle 8, 3, 2, 1\rangle\!\rangle$	.	1	.	1	.
$\star$	40768	$\langle\!\langle 6, 5, 3, +\rangle\!\rangle$	1	1	1	.	.
	40768	$\langle\!\langle 6, 5, 3, -\rangle\!\rangle$	1	1	1	.	.
	54912	$\langle\!\langle 6, 4, 3, 1\rangle\!\rangle$	2	1	.	.	.

Table 8.10: 5-modular decomposition matrix of block $\mathfrak{b}_{13}$ of $\tilde{\mathrm{A}}_{14}$
(defect 2, bar core $(3,1)$, weight 2, content $(6,5,3)$)

			$[\![5^2,3,1,+]\!]$ 7456	$[\![5^2,3,1,-]\!]$ 7456	$[\![6,4,3,1,+]\!]$ 12544	$[\![6,4,3,1,-]\!]$ 12544	$[\![6,5,3,+]\!]$ 384	$[\![6,5,3,-]\!]$ 384	$[\![8,3,2,1,+]\!]$ 7840	$[\![8,3,2,1,-]\!]$ 7840	$[\![8,5,1,+]\!]$ 6272	$[\![8,5,1,-]\!]$ 6272
$\star$	384	$\langle\!\langle 13,1,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	.
$\star$	384	$\langle\!\langle 13,1,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	.
$\star$	6656	$\langle\!\langle 11,3,+\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	1
$\star$	6656	$\langle\!\langle 11,3,-\rangle\!\rangle$	.	.	.	.	.	1	.	.	1	.
	28224	$\langle\!\langle 10,3,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	1	1
$\star$	13728	$\langle\!\langle 8,6,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	1	.
$\star$	13728	$\langle\!\langle 8,6,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	1
	68992	$\langle\!\langle 8,5,1\rangle\!\rangle$	1	1	1	1	1	1	1	1	1	1
$\star$	20384	$\langle\!\langle 8,3,2,1,+\rangle\!\rangle$	.	.	.	1	.	.	1	.	.	.
$\star$	20384	$\langle\!\langle 8,3,2,1,-\rangle\!\rangle$	.	.	1	.	.	.	.	1	.	.
	40768	$\langle\!\langle 6,5,3\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.
$\star$	27456	$\langle\!\langle 6,4,3,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.	.
$\star$	27456	$\langle\!\langle 6,4,3,1,-\rangle\!\rangle$	1	1	.	1	.	.	.	.	.	.

Table 8.11: 5-modular decomposition matrix of block b_{12} of $\tilde{A}_{14}$
(defect 2, bar core (4), weight 2, content $(5,6,3)$)

			$[\![5,4,3,2]\!]$ 4576	$[\![5^2,4]\!]$ 64	$[\![7,4,3]\!]$ 50624	$[\![8,4,2]\!]$ 40768	$[\![9,4,1]\!]$ 13664
$\star$	64	$\langle\!\langle 14\rangle\!\rangle$	.	1	.	.	.
$\star$	13728	$\langle\!\langle 10,4,+\rangle\!\rangle$	.	1	.	.	1
	13728	$\langle\!\langle 10,4,-\rangle\!\rangle$	.	1	.	.	1
	18304	$\langle\!\langle 9,5,+\rangle\!\rangle$	1	1	.	.	1
	18304	$\langle\!\langle 9,5,-\rangle\!\rangle$	1	1	.	.	1
	59136	$\langle\!\langle 9,4,1\rangle\!\rangle$	1	2	.	1	1
$\star$	40768	$\langle\!\langle 9,3,2\rangle\!\rangle$	.	.	.	1	.
	96096	$\langle\!\langle 8,4,2\rangle\!\rangle$	1	2	1	1	.
$\star$	59904	$\langle\!\langle 7,4,3\rangle\!\rangle$	2	2	1	.	.
$\star$	4576	$\langle\!\langle 5,4,3,2,+\rangle\!\rangle$	1	.	.	.	.
	4576	$\langle\!\langle 5,4,3,2,-\rangle\!\rangle$	1	.	.	.	.

Table 8.12: 5-modular decomposition matrix of block B_{21} of $\tilde{\mathrm{S}}_{14}$
(defect 2, bar core (4), weight 2, content $(5,6,3)$)

			$[\![5,4,3,2,+]\!]$ 4576	$[\![5,4,3,2,-]\!]$ 4576	$[\![5^2,4,+]\!]$ 64	$[\![5^2,4,-]\!]$ 64	$[\![7,4,3,+]\!]$ 50624	$[\![7,4,3,-]\!]$ 50624	$[\![8,4,2,+]\!]$ 40768	$[\![8,4,2,-]\!]$ 40768	$[\![9,4,1,+]\!]$ 13664	$[\![9,4,1,-]\!]$ 13664
$\star$	64	$\langle\!\langle 14,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.
$\star$	64	$\langle\!\langle 14,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
$\star$	27456	$\langle\!\langle 10,4\rangle\!\rangle$	.	.	1	1	.	.	.	.	1	1
	36608	$\langle\!\langle 9,5\rangle\!\rangle$	1	1	1	1	.	.	.	.	1	1
$\star$	59136	$\langle\!\langle 9,4,1,+\rangle\!\rangle$	1	.	1	1	.	.	1	.	.	1
	59136	$\langle\!\langle 9,4,1,-\rangle\!\rangle$	.	1	1	1	.	.	.	1	1	.
$\star$	40768	$\langle\!\langle 9,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.
$\star$	40768	$\langle\!\langle 9,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.
$\star$	96096	$\langle\!\langle 8,4,2,+\rangle\!\rangle$	1	.	1	1	.	1	1	.	.	.
	96096	$\langle\!\langle 8,4,2,-\rangle\!\rangle$	.	1	1	1	1	.	.	1	.	.
$\star$	59904	$\langle\!\langle 7,4,3,+\rangle\!\rangle$	1	1	1	1	1	.	.	.	.	.
$\star$	59904	$\langle\!\langle 7,4,3,-\rangle\!\rangle$	1	1	1	1	.	1	.	.	.	.
$\star$	9152	$\langle\!\langle 5,4,3,2\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.

9 The 7-modular spin characters of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

$\tilde{S}_{14}$ has nine spin 7-blocks of defect 0, and $\tilde{A}_{14}$ has six spin 7-blocks of defect 0. For all other spin 7-blocks of $\tilde{S}_{14}$ and $\tilde{A}_{14}$, we collect their constants in Table 9.1.

Table 9.1: Spin 7-blocks of non-zero defect of $\tilde{S}_{14}$ and $\tilde{A}_{14}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
54	17	9	2	$()$	2	28	22	18
57	7	6	1	$(4,2,1)$	1	30	5	3

The 7-modular basic spin character of $\tilde{S}_{14}$ is $\sigma(14) := \sigma\langle\langle 14,+\rangle\rangle|_{7'} = \sigma\langle\langle 14,-\rangle\rangle|_{7'}$. Then $\sigma(13,1) := \sigma\langle\langle 13,1\rangle\rangle|_{7'} - \sigma(14)$ is an irreducible Brauer character of $\tilde{S}_{14}$, see (5.5.4). To compute the 7-modular basic spin characters of $\tilde{A}_{14}$, we construct a basic spin representation of $\tilde{S}_{14}$ over GF(49) and apply (3.12). We obtain the irreducible constituents $\lambda(14)$ and $\lambda(14)^c$ of $\sigma(14)\downarrow_{\tilde{A}_{14}}$.

9.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{57} and b_{30} are given in Appendix A as Figures A.5 and A.6, respectively.

9.2 The block B_{54}

We start with basic sets $\{\sigma_1,\dots,\sigma_9\}$ and $\{\Sigma_1,\dots,\Sigma_9\}$ for B_{54} defined by Table 9.2.
Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1,2\}$. The mutual scalar products show that $\underline{\sigma}_3 := \sigma_3$ is a Brauer atom, thus irreducible, and $\underline{\Sigma}_i := \Sigma_i$ for $i \in \{1,5,6\}$ is a projective atom and therefore indecomposable, see Table 9.3.
As $\sigma_{10} = -\underline{\sigma}_3 + \sigma_4 + \sigma_7 - \sigma_9$, we can deduce from the decomposition of σ_4 in terms of the Brauer atoms that $\underline{\sigma}_4 := \sigma_4 - \underline{\sigma}_3$ is irreducible. Hence $\underline{\Sigma}_3 := \Sigma_3$ is indecomposable. Similarly, the relation $\sigma_{11} = \underline{\sigma}_1 - 2\underline{\sigma}_4 + 2\sigma_5 + \sigma_6 - 2\sigma_7 + 2\sigma_9$ shows that $\underline{\sigma}_5 := \sigma_5 - \underline{\sigma}_4$ is

Table 9.2: Some characters for B_{54}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(14)$	64	$\Sigma[\![7,6]\!]\uparrow^{\tilde{S}_{14}}$	119168
2	$\sigma(13,1)$	704	$\Sigma[\![8,5,-]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	68992
3	$\sigma\langle\!\langle 5,4,3,2\rangle\!\rangle\|_{7'}$	9152	$\Sigma[\![5,4,3,1,+]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	310464
4	$\sigma\langle\!\langle 6,4,3,1\rangle\!\rangle\|_{7'}$	54912	$\Sigma[\![4,3,2,1,\pm:3,1,\pm]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}/2$	407680
5	$\sigma\langle\!\langle 6,5,2,1\rangle\!\rangle\|_{7'}$	49920	$\Sigma[\![6,5:2,1]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	351232
6	$\sigma(14)\otimes\sigma[9,4,1]\|_{B_{54}}$	46592	$\Sigma[\![7,5,1,-]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	310464
7	$\sigma\langle\!\langle 10,4\rangle\!\rangle\|_{7'}$	27456	$\Sigma[\![7,4,2,+]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	366912
8	$\sigma\langle\!\langle 12,2\rangle\!\rangle\|_{7'}$	4160	$\Sigma[\![9,4,-]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	81536
9	$\sigma\langle\!\langle 11,3\rangle\!\rangle\|_{7'}$	13312	$\Sigma[\![10,3,+]\!]\uparrow^{\tilde{S}_{14}}\|_{B_{54}}$	40768
10	$\sigma\langle\!\langle 7,4,3,+\rangle\!\rangle\|_{7'}$	59904		
11	$\sigma(14)\otimes\sigma[7^2]$	26688		

Table 9.3: Mutual scalar products (B_{54})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9
$\underline{\sigma_1}$	1	.	.	.	.	.	.	.	.
$\underline{\sigma_2}$	.	1	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.
σ_4	.	.	1	1	.	.	.	.	.
σ_5	.	.	.	1	1	.	.	.	.
σ_6	.	.	.	.	.	1	2	.	.
σ_7	.	.	.	.	.	.	1	1	1
σ_8	.	1	.	.	.	.	.	1	.
σ_9	.	.	.	.	.	.	.	1	1

irreducible, and $\underline{\Sigma}_4 := \Sigma_4$ is indecomposable. Then $\underline{\Sigma}_i$ corresponds to $\underline{\sigma}_i$ for $i \in \{1,4,5\}$, and $\langle \sigma_{10}, \underline{\Sigma}_4 \rangle = 1$ implies that $\underline{\sigma}_4$ is contained in σ_{10}. Since $\underline{\sigma}_7 := \sigma_{10} - \underline{\sigma}_4 = \Sigma_7^*$ is a Brauer atom, we have found another irreducible Brauer character. Analogously, $\langle \sigma_{11}, \underline{\Sigma}_1 \rangle = 1$ and $\langle \sigma_{11}, \underline{\Sigma}_5 \rangle = 2$ lead to $\underline{\sigma}_6 := \sigma_{11} - \underline{\sigma}_1 - 2\underline{\sigma}_5$ which is irreducible. We exchange $\{\underline{\sigma}_6, \underline{\sigma}_7\}$ with $\{\sigma_6, \sigma_7\}$ and get the mutual scalar products in Table 9.4. Clearly, $\underline{\Sigma}_7 := \Sigma_7$ and $\underline{\Sigma}_9 := \Sigma_9$ are indecomposable. At this point, there are no more relations that give information on whether or not σ_8 or σ_9 are irreducible. However, by now we have enough information to calculate all irreducible Brauer characters of b_{28}, which in turn determine those of B_{54}.

Table 9.4: Mutual scalar products (B_{54})

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	Σ_2	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	Σ_7	Σ_8	Σ_9
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	.	1	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	.	1	.	.	.	.	.
$\underline{\sigma}_5$	.	.	.	.	1	.	.	.	.
$\underline{\sigma}_6$	.	.	.	.	.	1	.	.	.
$\underline{\sigma}_7$	.	.	.	.	.	.	1	.	.
σ_8	.	1	.	.	.	.	.	1	.
σ_9	.	.	.	.	.	.	.	1	1

9.3 The block b_{28}

The basic sets $\{\lambda_1, \ldots, \lambda_{18}\}$ and $\{\Lambda_1, \ldots, \lambda_{18}\}$ for b_{28} are defined by Table 9.5. Recall that $\underline{\lambda}_1 := \lambda_1$ and $\underline{\lambda}_2 := \lambda_2$ are irreducible. We see in Table 9.6 that $\underline{\lambda}_3 := \lambda_3$ and $\underline{\lambda}_4 := \lambda_4$ are Brauer atoms and therefore irreducible. Since

$$\underline{\sigma}_5(B_{54})\downarrow_{\tilde{A}_{14}} = \lambda_{10} = \Lambda_9^* + \Lambda_{10}^*,$$
$$\underline{\sigma}_6(B_{54})\downarrow_{\tilde{A}_{14}} = \lambda_{14} = \Lambda_{13}^* + \Lambda_{14}^*,$$

$\underline{\lambda}_9 := \Lambda_9^*$, $\underline{\lambda}_{10} := \Lambda_{10}^*$, $\underline{\lambda}_{13} := \Lambda_{13}^*$, and $\underline{\lambda}_{14} = \Lambda_{14}^*$ are irreducible Brauer characters. Moreover, it follows from the relation $\underline{\sigma}_2(B_{54})\downarrow_{\tilde{A}_{14}} = \lambda_5 + \lambda_6 - \underline{\lambda}_1 - \underline{\lambda}_2$ and Table 9.6 that $\underline{\lambda}_5 := \lambda_5 - \underline{\lambda}_2$ and $\underline{\lambda}_6 := \lambda_6 - \underline{\lambda}_1$ are irreducible.

Further relations $\lambda_{19} = \underline{\lambda}_1 + \underline{\lambda}_5 - \underline{\lambda}_6 + \lambda_7$, $\lambda_{20} = \underline{\lambda}_2 - \underline{\lambda}_5 + \underline{\lambda}_6 + \lambda_8$, $\lambda_9 = -\underline{\lambda}_3 + \underline{\lambda}_9 + \lambda_{18}$, and $\underline{\sigma}_4(B_{54}) \downarrow_{\tilde{A}_{14}} = -\underline{\lambda}_3 - \underline{\lambda}_{10} + \lambda_{17} + \lambda_{18}$ show that $\underline{\lambda}_7 := \lambda_7 - \underline{\lambda}_6$ and $\underline{\lambda}_8 := \lambda_8 - \underline{\lambda}_5$ are irreducible, and that $\underline{\lambda}_3$ and $\underline{\lambda}_{10}$ are contained in λ_{18} and λ_{17}, respectively. Hence $\lambda_{18a} := \lambda_{18} - \underline{\lambda}_3$ and $\lambda_{17a} := \lambda_{17} - \underline{\lambda}_{10}$ improve the basic set of Brauer characters. Then $\lambda_{21} = \underline{\lambda}_2 + \underline{\lambda}_6 - \underline{\lambda}_7 + \underline{\lambda}_8 + \lambda_{12}$ and $\lambda_{22} = \underline{\lambda}_1 + \underline{\lambda}_5 + \underline{\lambda}_7 - \underline{\lambda}_8 + \lambda_{11}$ lead to $\underline{\lambda}_{12} := \lambda_{12} - \underline{\lambda}_7$ and $\underline{\lambda}_{11} := \lambda_{11} - \underline{\lambda}_8$. As $\underline{\sigma}_7(B_{54})\downarrow_{\tilde{A}_{14}} = \Lambda^*_{15} + \Lambda^*_{16}$, the Brauer atoms $\underline{\lambda}_{15} := \Lambda^*_{15}$ and $\underline{\lambda}_{16} := \Lambda^*_{16}$ are irreducible Brauer characters. Finally, $\underline{\Sigma}_3(B_{54}) \downarrow_{\tilde{A}_{14}} = \Lambda_3 + \Lambda_4 - \Lambda_{17} - \Lambda_{18}$ implies that $\underline{\lambda}_{17} := \lambda_{17a}$ and $\underline{\lambda}_{18a} := \lambda_{18a}$ are irreducible (see Table 9.7) and we are done. In particular, $\underline{\lambda}_{17}\uparrow^{\tilde{S}_{14}} = \sigma_8 - \underline{\sigma}_2$ and $\underline{\lambda}_{18}\uparrow^{\tilde{S}_{14}} = \sigma_9 - \underline{\sigma}_8$ complete $\mathrm{IBr}(B_{54})$. The decomposition matrices of B_{54} and b_{28} are Tables 9.8 and 9.9, respectively.

Table 9.5: Some characters for b_{28}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(14)$	32	$\Lambda[\![7,6,+]\!]\uparrow^{\tilde{A}_{14}}$	59584
2	$\lambda(14)^c$	32	$\Lambda[\![7,6,-]\!]\uparrow^{\tilde{A}_{14}}$	59584
3	$\lambda\langle\!\langle 5,4,3,2,+\rangle\!\rangle\vert_{7'}$	4576	$\Lambda[\![4,3,2,+:5,-]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	359072
4	$\lambda\langle\!\langle 5,4,3,2,-\rangle\!\rangle\vert_{7'}$	4576	$\Lambda[\![4,3,2,+:5,+]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	359072
5	$\lambda\langle\!\langle 13,1,+\rangle\!\rangle\vert_{7'}$	384	$\lambda(14)\otimes\Lambda[12,2]$	34496
6	$\lambda\langle\!\langle 13,1,-\rangle\!\rangle\vert_{7'}$	384	$\lambda(14)^c\otimes\Lambda[12,2]$	34496
7	$\lambda\langle\!\langle 12,2,-\rangle\!\rangle\vert_{7'}$	2080	$\lambda(14)\otimes\Lambda[11,3]$	40768
8	$\lambda\langle\!\langle 12,2,+\rangle\!\rangle\vert_{7'}$	2080	$\lambda(14)^c\otimes\Lambda[10,2^2]\vert_{b_{28}}$	40768
9	$\lambda\langle\!\langle 6,5,2,1,+\rangle\!\rangle\vert_{7'}$	24960	$\Lambda[\![6,5,2,+]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	175616
10	$\underline{\sigma}_5(B_{54})\downarrow_{\tilde{A}_{14}}$	4160	$\Lambda[\![6,5,2,-]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	175616
11	$\lambda\langle\!\langle 11,3,+\rangle\!\rangle\vert_{7'}$	6656	$\lambda(14)\otimes\Lambda[10,4]$	20384
12	$\lambda\langle\!\langle 11,3,-\rangle\!\rangle\vert_{7'}$	6656	$\lambda(14)^c\otimes\Lambda[10,4]$	20384
13	$\lambda(14)\otimes\lambda\langle 3,2^3,1^5\rangle\vert_{7'}\vert_{b_{28}}$	52416	$\lambda(14)\otimes\Lambda[6^2,1^2]\vert_{b_{28}}$	155232
14	$\underline{\sigma}_6(B_{54})\downarrow_{\tilde{A}_{14}}$	18304	$\lambda(14)^c\otimes\Lambda[6^2,1^2]\vert_{b_{28}}$	155232
15	$\lambda\langle\!\langle 10,4,+\rangle\!\rangle\vert_{7'}$	13728	$\lambda(14)\otimes\Lambda[8,4,2]\vert_{b_{28}}$	183456
16	$\lambda\langle\!\langle 10,4,-\rangle\!\rangle\vert_{7'}$	13728	$\lambda(14)^c\otimes\Lambda[8,4,2]\vert_{b_{28}}$	183456
17	$\lambda\langle\!\langle 6,5,2,1,-\rangle\!\rangle\vert_{7'}$	24960	$\Lambda[\![6,4,3,+]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	203840
18	$\lambda\langle\!\langle 6,4,3,1,-\rangle\!\rangle\vert_{7'}$	27456	$\Lambda[\![6,4,3,-]\!]\uparrow^{\tilde{A}_{14}}\vert_{b_{28}}$	203840
19	$\lambda(14)\otimes\lambda[12,1^2]$	2112		
20	$\lambda(14)^c\otimes\lambda[12,1^2]$	2112		
21	$\lambda(14)\otimes\lambda[11,1^3]$	7040		
22	$\lambda(14)^c\otimes\lambda[11,1^3]$	7040		

Table 9.6: Mutual scalar products (b_{28})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_4	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_5	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_6	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_7	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.
λ_8	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.
λ_9	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	1
λ_{10}	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
λ_{11}	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
λ_{12}	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.
λ_{13}	.	.	.	.	1	.	2	2	.	.	1	2	1	.	1	2	.	.
λ_{14}	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
λ_{15}	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.
λ_{16}	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
λ_{17}	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.
λ_{18}	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Table 9.7: Mutual scalar products (b_{28})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_5$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_7$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_8$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_9$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_{10}$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
λ_{11}	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
λ_{12}	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.
$\underline{\lambda}_{13}$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
$\underline{\lambda}_{14}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
λ_{15}	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.
λ_{16}	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
λ_{17a}	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
λ_{18a}	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Table 9.8: 7-modular decomposition matrix of block B_{54} of $\tilde{S}_{14}$
(defect 2, bar core $()$, weight 2, content $(4,4,4,2)$)

			$[\![5,4,3,2]\!]$ 9152	$[\![6,4,3,1]\!]$ 45760	$[\![6,5,2,1]\!]$ 4160	$[\![7,4,3]\!]$ 14144	$[\![7,5,2]\!]$ 18304	$[\![7,6,1]\!]$ 64	$[\![8,6]\!]$ 704	$[\![9,5]\!]$ 3456	$[\![10,4]\!]$ 9856
$\star$	64	$\langle\langle 14,+\rangle\rangle$	.	.	.	.	.	1	.	.	.
	64	$\langle\langle 14,-\rangle\rangle$	.	.	.	.	.	1	.	.	.
$\star$	768	$\langle\langle 13,1\rangle\rangle$	.	.	.	.	.	1	1	.	.
$\star$	4160	$\langle\langle 12,2\rangle\rangle$	.	.	.	.	.	.	1	1	.
$\star$	13312	$\langle\langle 11,3\rangle\rangle$	.	.	.	.	.	.	.	1	1
$\star$	27456	$\langle\langle 10,4\rangle\rangle$	.	.	.	1	.	.	.	1	1
	36608	$\langle\langle 9,5\rangle\rangle$	.	.	.	1	1	.	1	1	.
$\star$	27456	$\langle\langle 8,6\rangle\rangle$	.	.	2	.	1	2	1	.	.
$\star$	31680	$\langle\langle 7,6,1,+\rangle\rangle$	1	.	1	.	1	1	.	.	.
	31680	$\langle\langle 7,6,1,-\rangle\rangle$	1	.	1	.	1	1	.	.	.
	91520	$\langle\langle 7,5,2,+\rangle\rangle$	1	1	1	1	1	.	.	.	.
	91520	$\langle\langle 7,5,2,-\rangle\rangle$	1	1	1	1	1	.	.	.	.
	59904	$\langle\langle 7,4,3,+\rangle\rangle$	.	1	.	1	.	.	.	.	.
	59904	$\langle\langle 7,4,3,-\rangle\rangle$	.	1	.	1	.	.	.	.	.
$\star$	49920	$\langle\langle 6,5,2,1\rangle\rangle$	.	1	1	.	.	.	.	.	.
	54912	$\langle\langle 6,4,3,1\rangle\rangle$	1	1	.	.	.	.	.	.	.
$\star$	9152	$\langle\langle 5,4,3,2\rangle\rangle$	1	.	.	.	.	.	.	.	.

Table 9.9: 7-modular decomposition matrix of block b_{28} of $\tilde{A}_{14}$
(defect 2, bar core $()$, weight 2, content $(4,4,4,2)$)

			⟦5,4,3,2,+⟧ 4576	⟦5,4,3,2,−⟧ 4576	⟦6,4,3,1,+⟧ 22880	⟦6,4,3,1,−⟧ 22880	⟦6,5,2,1,+⟧ 2080	⟦6,5,2,1,−⟧ 2080	⟦7,4,3,+⟧ 7072	⟦7,4,3,−⟧ 7072	⟦7,5,2,+⟧ 9152	⟦7,5,2,−⟧ 9152	⟦7,6,1,+⟧ 32	⟦7,6,1,−⟧ 32	⟦8,6,+⟧ 352	⟦8,6,−⟧ 352	⟦9,5,+⟧ 1728	⟦9,5,−⟧ 1728	⟦10,4,+⟧ 4928	⟦10,4,−⟧ 4928
	64	⟪14⟫	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
⋆	384	⟪13,1,+⟫	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
	384	⟪13,1,−⟫	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	2080	⟪12,2,+⟫	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
	2080	⟪12,2,−⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
⋆	6656	⟪11,3,+⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
	6656	⟪11,3,−⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
⋆	13728	⟪10,4,+⟫	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	1
⋆	13728	⟪10,4,−⟫	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	1	.
⋆	18304	⟪9,5,+⟫	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	1	.	.
⋆	18304	⟪9,5,−⟫	.	.	.	.	.	.	1	.	1	.	.	.	.	1	1	.	.	.
⋆	13728	⟪8,6,+⟫	.	.	.	.	1	1	.	.	.	1	1	1	1	.	.	.	.	.
⋆	13728	⟪8,6,−⟫	.	.	.	.	1	1	.	.	1	.	1	1	.	1	.	.	.	.
⋆	31680	⟪7,6,1⟫	1	1	.	.	1	1	.	.	1	1	1	1	.	.	.	.	.	.
⋆	91520	⟪7,5,2⟫	1	1	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.
⋆	59904	⟪7,4,3⟫	.	.	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.
⋆	24960	⟪6,5,2,1,+⟫	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	24960	⟪6,5,2,1,−⟫	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	27456	⟪6,4,3,1,+⟫	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	27456	⟪6,4,3,1,−⟫	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	4576	⟪5,4,3,2,+⟫	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	4576	⟪5,4,3,2,−⟫	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

10 The 3-modular spin characters of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

There is one spin 3-block of defect 0 of $\tilde{S}_{15}$ that covers two such blocks of $\tilde{A}_{15}$. The block constants of all other spin 3-blocks of $\tilde{S}_{15}$ and $\tilde{A}_{15}$ are given in Table 10.1.

Table 10.1: Spin 3-blocks of non-zero defect of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
8	36	14	6	$()$	5	5	36	7
9	3	1	1	$(7,4,1)$	1	6	3	2

We compute the 3-modular basic spin character $\sigma(15)$ of $\tilde{S}_{15}$ *directly*, that is, we construct a basic spin representation T of $\tilde{S}_{15}$ over $GF(9)$ and compute the Brauer character value of t_λ^T for all class representatives t_λ in question. Then $\sigma\langle\langle 15\rangle\rangle|_{3'} = \sigma(15) + \sigma(15)^a$ and the 3-modular basic spin character of $\tilde{A}_{15}$ is $\lambda(15) := \sigma(15)\downarrow_{\tilde{A}_{15}}$. By (5.5.4), either $\sigma(15)$ or $\sigma(15)^a$ is contained in $\sigma\langle\langle 14,1,+\rangle\rangle|_{5'}$ but we cannot decide which one yet. Nevertheless, $\underline{\lambda}(14,1) := \lambda\langle\langle 14,1\rangle\rangle|_{3'} - \lambda(15)$ is an irreducible Brauer character of $\tilde{A}_{15}$.

10.1 Spin blocks of defect 1

The Brauer trees of the blocks B_9 and b_6 are given in Appendix A as Figures A.7 and A.8, respectively.

10.2 The block b_5

We begin with a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1,\dots,7\}$ defined by Table 10.2. Their mutual scalar products are given in Table 10.3. Clearly, $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1,2,3\}$ and $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{4,6,7\}$.

Table 10.2: Some characters for $\mathfrak{b}_5$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(15)$	64	$\Lambda[\![3^4, 2]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	20808576
2	$\underline{\lambda}(14, 1)$	768	$\Lambda[\![4, 3^3, 1, +]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	6718464
3	$\lambda\langle\!\langle 5, 4, 3, 2, 1, -\rangle\!\rangle\|_{3'}$	4576	$\Lambda[\![5, 4, 3, 2]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	10567584
4	$\lambda\langle\!\langle 13, 2\rangle\!\rangle\|_{3'}$	4928	$\Lambda[\![5, 3^2, 2, 1]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}/2$	3872448
5	$\lambda\langle\!\langle 8, 7\rangle\!\rangle\|_{3'}$	27456	$\Sigma[\![6, 5, 2, -]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	4432320
6	$\lambda(15) \otimes \lambda[11, 4]$	35840	$\Lambda[\![6, 4, 3, 1, -]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	3032640
7	$\lambda\langle\!\langle 8, 6, 1, -\rangle\!\rangle\|_{3'}$	114400	$\Lambda[\![7, 4, 2, 1, +]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	2309472
8	$\lambda[\![5, 3^2, 2, 1]\!]\uparrow^{\tilde{A}_{15}}$	30240		
9	$\lambda[\![5, 4, 3, 2]\!]\uparrow^{\tilde{A}_{15}}$	68640		
10	$\lambda[\![6, 4, 3, 1, -]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	99072		
11	$\lambda[\![7, 5, 2]\!]\uparrow^{\tilde{A}_{15}}\|_{\mathfrak{b}_5}$	246144		

Table 10.3: Mutual scalar products ($\mathfrak{b}_5$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7
λ_1	1	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.
λ_3	.	.	1	.	.	.	.
λ_4	2	1	.	1	.	.	.
λ_5	2	.	.	.	1	.	.
λ_6	.	.	2	.	.	1	.
λ_7	3	1	1	.	1	.	1

Table 10.4: Mutual scalar products ($\mathfrak{b}_5$)

$\langle\cdot,\cdot\rangle$	Λ_1	$\underline{\Lambda}_2$	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	$\underline{\Lambda}_5$	$\underline{\Lambda}_6$	$\underline{\Lambda}_7$
$\underline{\lambda}_1$	1	.	.	.	.	.	.
λ_{4a}	1	.	.	1	.	.	.
λ_{5a}	1	.	.	.	1	.	.

As

$$\begin{aligned}
\lambda_8 &= -2\underline{\lambda}_1 - 4\underline{\lambda}_2 + 3\underline{\lambda}_3 + 4\lambda_4,\\
\lambda_{10} &= -\underline{\lambda}_1 + \lambda_5 + 2\lambda_6,\\
\lambda_9 &= 2\underline{\lambda}_2 - \underline{\lambda}_3 + 2\lambda_6,\\
\lambda_{11} &= -3\underline{\lambda}_1 - \underline{\lambda}_2 - 3\underline{\lambda}_3 - 3\lambda_5 + 3\lambda_7,
\end{aligned}$$

we can improve the basic set of Brauer characters by exchanging $\lambda_4, \dots, \lambda_7$ with

$$\begin{aligned}
\lambda_{4a} &:= \lambda_4 - \underline{\lambda}_1 - \underline{\lambda}_2,\\
\lambda_{5a} &:= \lambda_5 - \underline{\lambda}_1,\\
\lambda_{6a} &:= \lambda_6 - \underline{\lambda}_3,\\
\lambda_{7a} &:= \lambda_7 - \underline{\lambda}_2 - \underline{\lambda}_3 - \lambda_5 + \underline{\lambda}_1.
\end{aligned}$$

To justify the improvement of λ_7, note that λ_5 has an irreducible constituent $\underline{\lambda}_5$ of the form $x\Lambda_1^* + \Lambda_5^*$ for a suitable $x \in \{0, 1, 2\}$. In particular, $\underline{\lambda}_1 = \Lambda_1^*$ could be contained twice in λ_5. Since the decomposition of λ_{11} in terms of the basic set shows that $\underline{\lambda}_5$ is contained in λ_7, we may subtract λ_5 from λ_7 if we add $2\underline{\lambda}_1$.
Then $\lambda_{11} = -6\underline{\lambda}_1 + 2\underline{\lambda}_2 + 3\lambda_{7a}$ shows that indeed we can improve λ_{7a} by $\underline{\lambda}_7 := \lambda_{7a} - 2\underline{\lambda}_1$ which is now irreducible. Moreover, the relation $\underline{\lambda}_3 \otimes \lambda[14, 1] = 2\underline{\lambda}_2 - \underline{\lambda}_3 + 2\lambda_{6a}$ leads to $\underline{\lambda}_6 := \lambda_{6a} - \underline{\lambda}_3$ which is irreducible too. Set $\underline{\Lambda}_i := \Lambda_i$ for $i \in \{2, 3, 5\}$. The yet open questions whether or not $\underline{\lambda}_1$ is contained in $\lambda \in \{\lambda_{4a}, \lambda_{5a}\}$ can be deduced from Table 10.4. We collect the degrees of the elements of the basic set of Brauer characters in Table 10.5.

Table 10.5: Degrees of basic set characters of b_5

φ	$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	λ_{4a}	λ_{5a}	$\underline{\lambda}_6$	$\underline{\lambda}_7$
$\deg \varphi$	64	768	4576	4096	27392	26688	81536

10.3 The block B_8

Although we have not determined $\mathrm{IBr}(b_5)$ completely, we start the computations for B_8 of $\tilde{S}_{15}$ which covers b_5. A basic set of Brauer characters σ_i and a basic set of projective characters Σ_i for $i \in \{1, \ldots, 18\}$ are defined by Table 10.6.

Table 10.6: Some characters for B_8

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(15)$	64	$\Sigma[\![3^4, 2, -]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	20808576
2	$\sigma(15)^a$	64	$\Sigma[\![3^4, 2, +]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	20808576
3	$\sigma\langle\!\langle 14, 1, -\rangle\!\rangle\vert_{3'}$	832	$\Sigma[\![4, 3^3, 1]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	13436928
4	$\sigma\langle\!\langle 14, 1, +\rangle\!\rangle\vert_{3'}$	832	$\sigma(15) \otimes \Sigma[10, 2^2, 1]\vert_{B_8}$	19082304
5	$\sigma\langle\!\langle 13, 2, -\rangle\!\rangle\vert_{3'}$	4928	$\sigma(15) \otimes \Sigma[6, 5, 2, 1^2]\vert_{B_8}$	20575296
6	$\sigma\langle\!\langle 13, 2, +\rangle\!\rangle\vert_{3'}$	4928	$\Sigma[\![5, 3^2, 2, 1, +]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	7744896
7	$\sigma\langle\!\langle 5, 4, 3, 2, 1\rangle\!\rangle\vert_{3'}$	9152	$\Sigma[\![5, 4, 3, 2, +]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	10567584
8	$\sigma[\![3^3, 2, 1 : 2, 1, -]\!]\uparrow^{\tilde{S}_{15}}$	14560	$\Sigma[\![5, 4, 3, 2, -]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	10567584
9	$\sigma\langle\!\langle 8, 7, -\rangle\!\rangle\vert_{3'}$	27456	$\Sigma[\![6, 5, 2, 1, +]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	8864640
10	$\sigma\langle\!\langle 8, 7, +\rangle\!\rangle\vert_{3'}$	27456	$\sigma(15) \otimes \Sigma[4, 3^2, 2^2, 1]\vert_{B_8}$	18289152
11	$\sigma(15) \otimes \sigma[11, 4]$	35840	$\underline{\sigma}(15) \otimes \Sigma[4^2, 3^2, 1]\vert_{B_8}$	7464960
12	$\sigma(15) \otimes \sigma[6, 5, 2^2]$	35840	$\Sigma[\![6, 4, 3, 1]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	6065280
13	$\lambda_7(b_5)\uparrow^{\tilde{S}_{15}}$	163072	$\Sigma[\![7, 5, 2, +]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	2309472
14	$\sigma[\![6, 5, 2, 1, -]\!]\uparrow^{\tilde{S}_{15}}$	204960	$\Sigma[\![7, 5, 2, -]\!]\uparrow^{\tilde{S}_{15}}\vert_{B_8}$	2309472
15	$\sigma(15) \otimes \sigma[13, 1^2]$	4992		
16	$\sigma(15) \otimes \sigma[7, 6, 2]$	4992		
17	$\sigma(15) \otimes \sigma[10, 5]$	99072		
18	$\sigma(15) \otimes \sigma[5^2, 3, 2]$	99072		

Set $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$. By (5.5.4), and Table 10.7, $\underline{\sigma}_3 := \sigma_3 - \underline{\sigma}_2$ and $\underline{\sigma}_4 := \sigma_4 - \underline{\sigma}_1$ are irreducible. It is also clear that $\underline{\Sigma}_{14} := \Sigma_{14}$ is indecomposable. Its associate character is $\underline{\Sigma}_{13} := \Sigma_{13}$. As $\underline{\lambda}_3(b_5)\uparrow^{\tilde{S}_{15}} = \sigma_7 = \Sigma_7^* + \Sigma_8^*$ has exactly two irreducible and associate constituents, these are $\underline{\sigma}_7 := \Sigma_7^*$ and $\underline{\sigma}_8 := \Sigma_8^*$. After substituting $\underline{\sigma}_i$ for σ_i ($i \in \{3, 4, 7, 8\}$), similar relations $\underline{\Lambda}_4(b_5)\uparrow^{\tilde{S}_{15}} = \Sigma_6 = \sigma_5^* + \sigma_6^*$ and $\underline{\Lambda}_6(b_5)\uparrow^{\tilde{S}_{15}} = \Sigma_{12} = \sigma_{11}^* + \sigma_{12}^*$ lead to projective indecomposable characters $\underline{\Sigma}_i := \sigma_i^*$ for $i \in \{5, 6, 11, 12\}$. We insert these in our basic set of projective characters and remove Σ_i for $i \in \{5, 6, 10, 12\}$. Furthermore, $\underline{\Lambda}_3(b_5)\uparrow^{\tilde{S}_{15}} = \Sigma_7 + \Sigma_8$ shows that $\underline{\Sigma}_7 := \Sigma_7$ and $\underline{\Sigma}_8 := \Sigma_8$ are indecomposable too. The new matrix of mutual scalar products is Table 10.8.

We can use the pairs $(\underline{\sigma}_7, \underline{\Sigma}_7)$ and $(\underline{\sigma}_8, \underline{\Sigma}_8)$ to improve the basic set of Brauer characters

Table 10.7: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	.	1	1	.	.	.	.	.	.	.	.	.	.	.
σ_4	1	.	1	1	.	.	.	.	.	.	.	.	.	.
σ_5	1	1	1	1	1	1	.	.	.	.	.	.	.	.
σ_6	1	1	1	2	.	1	.	.	.	.	.	.	.	.
σ_7	.	.	.	.	.	.	1	1	.	.	.	.	.	.
σ_8	3	3	2	3	1	2	.	1	.	.	.	.	.	.
σ_9	1	1	.	.	1	.	.	.	1	1	1	.	.	.
σ_{10}	1	1	.	.	.	.	.	.	1	.	.	.	.	.
σ_{11}	.	.	.	.	1	.	1	1	.	.	1	1	.	.
σ_{12}	.	.	.	.	.	.	1	1	.	.	.	1	.	.
σ_{13}	.	.	.	2	4	.	.	.	.	6	.	.	1	1
σ_{14}	3	3	.	1	4	.	1	2	4	5	2	.	1	.

Table 10.8: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	Σ_9	Σ_{11}	$\underline{\Sigma}_{11}$	$\underline{\Sigma}_{12}$	$\underline{\Sigma}_{13}$	$\underline{\Sigma}_{14}$
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	1	1	.	.	.	.	.	.	.	.	.	.
σ_5	1	1	1	1	1	.	.	.	.	.	.	.	.	.
σ_6	1	1	1	2	.	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_7$	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$\underline{\sigma}_8$	.	.	.	.	.	.	.	1	.	.	.	.	.	.
σ_9	1	1	.	.	.	.	.	.	1	1	.	.	.	.
σ_{10}	1	1	.	.	.	.	.	.	1	.	.	.	.	.
σ_{11}	.	.	.	.	.	.	1	1	.	1	1	.	.	.
σ_{12}	.	.	.	.	.	.	1	1	.	.	.	1	.	.
σ_{13}	.	.	.	2	.	.	.	.	.	.	.	.	1	1
σ_{14}	3	3	.	1	.	.	1	2	4	2	.	.	1	.

by exchanging σ_{11}, σ_{12} and σ_{14} with

$$\sigma_{11a} := \sigma_{11} - \underline{\sigma}_7 - \underline{\sigma}_8,$$
$$\sigma_{12a} := \sigma_{12} - \underline{\sigma}_7 - \underline{\sigma}_8,$$
$$\sigma_{14a} := \sigma_{14} - \underline{\sigma}_7 - 2\underline{\sigma}_8.$$

Then $\underline{\lambda}_6(b_5)\uparrow^{\tilde{S}_{15}} = \sigma_{11a} + \sigma_{12a}$ proves that $\underline{\sigma}_i := \sigma_{ia}$ is irreducible for $i \in \{11, 12\}$. As $\underline{\lambda}_7(b_5)\uparrow^{\tilde{S}_{15}} = \sigma_{13} = 2\Sigma_4^* + \underline{\Sigma}_{13}^* + \underline{\Sigma}_{14}^*$ with $\deg \Sigma_4^* = 0$ and $\deg \Sigma_{13}^* = \deg \Sigma_{14}^* = \deg \underline{\lambda}_7$, we see that the irreducible constituents of σ_{13} are $x\Sigma_4^* + \underline{\Sigma}_{13}^*$ and its associate character $(2-x)\Sigma_4^* + \underline{\Sigma}_{14}^*$ for a suitable $x \in \{0, 1, 2\}$. As is seen in Table 10.5, each pair of associate characters $\{\sigma, \sigma^a\} \subset \mathrm{IBr}(B_8)$ can be uniquely identified by $\deg \sigma$. In particular, there is only one pair of degree 26688 in $\mathrm{IBr}(B_8)$. This requires

$$(x\Sigma_4^* + \underline{\Sigma}_{13}^*)^{\circledast} \in \{x\Sigma_4^* + \underline{\Sigma}_{13}^*, (2-x)\Sigma_4^* + \underline{\Sigma}_{14}^*\}$$

which is fulfilled only for $x = 1$. Thus $\underline{\sigma}_{13} := \Sigma_4^* + \underline{\Sigma}_{13}^*$ and $\underline{\sigma}_{14} := \Sigma_4^* + \underline{\Sigma}_{14}^*$ are irreducible Brauer characters that we substitute for σ_{13} and σ_{14}, respectively. Furthermore, it follows that $\underline{\Sigma}_9 := \sigma_9^*$ and $\underline{\Sigma}_{10} := \sigma_{10}^*$ are projective indecomposable characters that we may exchange with Σ_9 and Σ_{11}, respectively. Then we can deduce from Table 10.9 that Σ_4 contains $\underline{\Sigma}_i$ for $i \in \{13, 14\}$ and we take $\Sigma_{4a} := \Sigma_4 - \underline{\Sigma}_{13} - \underline{\Sigma}_{14}$ instead of Σ_4.

Table 10.9: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	$\underline{\Sigma}_{11}$	$\underline{\Sigma}_{12}$	$\underline{\Sigma}_{13}$	$\underline{\Sigma}_{14}$
σ_5	1	1	1	1	1	.	.	.	.	.	.	.	.	.
σ_6	1	1	1	2	.	1	.	.	.	.	.	.	.	.
σ_9	1	1	.	.	.	.	.	.	1	.	.	.	.	.
σ_{10}	1	1	.	.	.	.	.	.	.	1	.	.	.	.
$\underline{\sigma}_{14}$	.	.	.	1	.	.	.	.	.	.	.	.	1	.
$\underline{\sigma}_{13}$	.	.	.	1	.	.	.	.	.	.	.	.	.	1

The relations

$$\sigma_{15} = \underline{\sigma}_1 + \underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_5,$$
$$\sigma_{16} = \underline{\sigma}_2 - \underline{\sigma}_3 + \underline{\sigma}_4 + \sigma_6$$

allow us to improve the basic set of Brauer characters by $\sigma_{5a} := \sigma_5 - \underline{\sigma}_4$ and $\sigma_{6a} := \sigma_6 - \underline{\sigma}_3$. Then $\underline{\Lambda}_2(b_5)\uparrow^{\tilde{S}_{15}} = \Sigma_2 = \sigma_3^* + \sigma_4^*$ gives the projective indecomposable characters $\underline{\Sigma}_3 := \sigma_3^*$ and $\underline{\Sigma}_4 := \sigma_4^*$. As

$$\sigma_{17} = -\underline{\sigma}_2 + 2\underline{\sigma}_7 + 2\underline{\sigma}_8 + \sigma_{10} + 2\underline{\sigma}_{12},$$
$$\sigma_{18} = -\underline{\sigma}_1 + 2\underline{\sigma}_7 + 2\underline{\sigma}_8 + \sigma_9 + 2\underline{\sigma}_{11},$$

we exchange σ_9 and σ_{10} with $\sigma_{9a} := \sigma_9 - \underline{\sigma}_1$ and $\sigma_{10a} := \sigma_6 - \underline{\sigma}_2$, respectively. Recall that we do not know whether or not λ_{4a} or λ_{5a} are irreducible. Since $\deg \sigma_{5a} = 4160$, we see in Table 10.5 that at least one element of $\{\underline{\sigma}_1, \underline{\sigma}_2\}$ is contained in σ_{5a}. Moreover, the (ir)reducibility of λ_{5a} corresponds to (ir)reducibility of σ_{9a} and σ_{10a} but we cannot make further deductions from the mutual scalar products (Table 10.10) or the relations that we are left with.

Table 10.10: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	$\underline{\Sigma}_{11}$	$\underline{\Sigma}_{12}$	$\underline{\Sigma}_{13}$	$\underline{\Sigma}_{14}$
σ_{5a}	1	1	.	.	1	.	.	.	.	.	.	.	.	.
σ_{6a}	1	1	.	.	.	1	.	.	.	.	.	.	.	.
σ_{9a}	.	1	.	.	.	.	.	.	1	.	.	.	.	.
σ_{10a}	1	.	.	.	.	.	.	.	.	1	.	.	.	.

Let $13a^+$ denote the irreducible reduced permutation module of S_{15} over GF(3), and let $64a^-$ be the basic spin module of $\tilde{S}_{15}$ over $F :=$ GF(9) that affords $\underline{\sigma}_1$. Chopping the sequence $(13a^+ \otimes 13a^+, 64a^- \otimes 78a^+)$ produces an irreducible $F\tilde{S}_{15}$-module $4096a^-$ whose restriction to $F\tilde{A}_{15}$ affords λ_{4a}. Thus $\lambda_{4a} \in \mathrm{IBr}(b_5)$. The trace $\mathrm{tr}_{4096a}(t_{(10,5)}) = 0_F$ suffices to identify the corresponding pair of irreducible Brauer characters for B_8, namely $\sigma_{5a} - \underline{\sigma}_1$ and $\sigma_{6a} - \underline{\sigma}_2$.

To resolve the question whether or not λ_{5a} respectively σ_{9a} and σ_{10a} are irreducible, we condense the tensor product $V := 64a^- \otimes 1287a^+$ with respect to the trivial idempotent $e := e_K$ of $K := \langle t_{(7,1^8)}, t_{(7,7,1)} \rangle \leqslant \tilde{S}_{15}$. Here $1287a^+$ is an irreducible constituent of $64a^- \otimes 64a^-$ with Brauer character $\sigma[10,3,2]$. The Brauer character of V decomposes as

$$\underline{\sigma}_1 \otimes \sigma[10,3,2] = 4\underline{\sigma}_2 + 2\underline{\sigma}_4 + 2\underline{\sigma}_6 + 2\underline{\sigma}_7 + 2\underline{\sigma}_8 + \sigma_{10a} + \underline{\sigma}_{11}.$$

Let S_i denote an $F\tilde{S}_{15}$-module that affords $\underline{\sigma}_i$ and put $H := eF\tilde{S}_{15}e$. Then the dimensions $\langle \underline{\sigma}_i{\downarrow_K}, 1_K \rangle$ of the irreducible H-modules $S_i e$ are given in Table 10.11.

Table 10.11: Dimensions of $eF\tilde{S}_{15}e$-modules

i	1	2	4	6	7	8	11
$\dim S_i e$	4	4	24	100	88	88	528

Let $a := t_1$, $b := t_{(15)}^{-1}$ and $c := (ab)^3 b$. With respect to the condensation algebra $C := \langle ece, e(ab)^2 bab^2 ce \rangle$, the condensed module Ve has an irreducible C-constituent X of dimension 548. Since $\langle \sigma_{10a}{\downarrow_K}, 1_K \rangle = 548$, X is a genuine H-module that corresponds to an irreducible $F\tilde{S}_{15}$-module with Brauer character σ_{10a}. Thus we have $\sigma_{9a}, \sigma_{10a} \in \mathrm{IBr}(B_8)$ and $\lambda_{5a} \in \mathrm{IBr}(b_5)$. The decomposition matrices of b_5 and B_8 are Tables 10.12 and 10.13, respectively.

Table 10.12: 3-modular decomposition matrix of block $\mathfrak{b}_5$ of $\tilde{A}_{15}$ (defect 6, bar core (), weight 5, content $(10, 5)$)

			$[\![3^4,2,1]\!]$ 64	$[\![4,3^3,2]\!]$ 768	$[\![5,3^3,1]\!]$ 4096	$[\![5,4,3,2,1]\!]$ 4576	$[\![6,4,3,2]\!]$ 26688	$[\![6,5,3,1]\!]$ 27392	$[\![7,5,2,1]\!]$ 81536
⋆	64	$\langle\!\langle 15,+\rangle\!\rangle$	1	.	.	.	.	.	.
	64	$\langle\!\langle 15,-\rangle\!\rangle$	1	.	.	.	.	.	.
⋆	832	$\langle\!\langle 14,1\rangle\!\rangle$	1	1	.	.	.	.	.
⋆	4928	$\langle\!\langle 13,2\rangle\!\rangle$	1	1	1	.	.	.	.
	17472	$\langle\!\langle 12,3\rangle\!\rangle$	2	.	2	2	.	.	.
	8800	$\langle\!\langle 12,2,1,+\rangle\!\rangle$	2	.	1	1	.	.	.
	8800	$\langle\!\langle 12,2,1,-\rangle\!\rangle$	2	.	1	1	.	.	.
⋆	40768	$\langle\!\langle 11,4\rangle\!\rangle$	1	1	1	2	1	.	.
	41600	$\langle\!\langle 11,3,1,+\rangle\!\rangle$	2	2	1	2	1	.	.
	41600	$\langle\!\langle 11,3,1,-\rangle\!\rangle$	2	2	1	2	1	.	.
	64064	$\langle\!\langle 10,5\rangle\!\rangle$	1	1	.	2	1	1	.
	68992	$\langle\!\langle 10,3,2,+\rangle\!\rangle$	2	2	1	2	1	1	.
	68992	$\langle\!\langle 10,3,2,-\rangle\!\rangle$	2	2	1	2	1	1	.
	64064	$\langle\!\langle 9,6\rangle\!\rangle$	2	.	.	2	.	2	.
	146432	$\langle\!\langle 9,5,1,+\rangle\!\rangle$	2	2	.	2	1	1	1
	146432	$\langle\!\langle 9,5,1,-\rangle\!\rangle$	2	2	.	2	1	1	1
	196000	$\langle\!\langle 9,4,2,+\rangle\!\rangle$	6	2	2	5	1	2	1
	196000	$\langle\!\langle 9,4,2,-\rangle\!\rangle$	6	2	2	5	1	2	1
	81536	$\langle\!\langle 9,3,2,1\rangle\!\rangle$	4	.	2	4	.	2	.
⋆	27456	$\langle\!\langle 8,7\rangle\!\rangle$	1	.	.	.	.	1	.
⋆	114400	$\langle\!\langle 8,6,1,+\rangle\!\rangle$	2	1	.	1	.	1	1
	114400	$\langle\!\langle 8,6,1,-\rangle\!\rangle$	2	1	.	1	.	1	1
	156000	$\langle\!\langle 8,4,3,+\rangle\!\rangle$	4	3	1	3	1	1	1
	156000	$\langle\!\langle 8,4,3,-\rangle\!\rangle$	4	3	1	3	1	1	1
	224224	$\langle\!\langle 8,4,2,1\rangle\!\rangle$	6	4	2	5	2	2	1
	123200	$\langle\!\langle 7,6,2,+\rangle\!\rangle$	4	1	1	2	.	1	1
	123200	$\langle\!\langle 7,6,2,-\rangle\!\rangle$	4	1	1	2	.	1	1
	192192	$\langle\!\langle 7,5,3,+\rangle\!\rangle$	6	3	2	4	1	2	1
	192192	$\langle\!\langle 7,5,3,-\rangle\!\rangle$	6	3	2	4	1	2	1
	228800	$\langle\!\langle 7,5,2,1\rangle\!\rangle$	6	4	2	6	2	2	1
	40768	$\langle\!\langle 6,5,4,+\rangle\!\rangle$	2	.	1	2	.	1	.
	40768	$\langle\!\langle 6,5,4,-\rangle\!\rangle$	2	.	1	2	.	1	.
	145600	$\langle\!\langle 6,5,3,1\rangle\!\rangle$	4	2	2	6	2	2	.
	64064	$\langle\!\langle 6,4,3,2\rangle\!\rangle$	.	2	.	2	2	.	.
⋆	4576	$\langle\!\langle 5,4,3,2,1,+\rangle\!\rangle$	.	.	.	1	.	.	.
	4576	$\langle\!\langle 5,4,3,2,1,-\rangle\!\rangle$	.	.	.	1	.	.	.

Table 10.13: 3-modular decomposition matrix of block B_8 of $\tilde{S}_{15}$ (defect 6, bar core $()$, weight 5, content $(10, 5)$)

			$[\![3^4,2,1,+]\!]$ 64	$[\![3^4,2,1,-]\!]$ 64	$[\![4,3^3,2,+]\!]$ 768	$[\![4,3^3,2,-]\!]$ 768	$[\![5,3^3,1,+]\!]$ 4096	$[\![5,3^3,1,-]\!]$ 4096	$[\![5,4,3,2,1,+]\!]$ 4576	$[\![5,4,3,2,1,-]\!]$ 4576	$[\![6,4,3,2,+]\!]$ 26688	$[\![6,4,3,2,-]\!]$ 26688	$[\![6,5,3,1,+]\!]$ 27392	$[\![6,5,3,1,-]\!]$ 27392	$[\![7,5,2,1,+]\!]$ 81536	$[\![7,5,2,1,-]\!]$ 81536
⋆	128	$\langle\!\langle 15\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	832	$\langle\!\langle 14,1,+\rangle\!\rangle$	1	.	.	1	.	.	.	.	.	.	.	.	.	.
⋆	832	$\langle\!\langle 14,1,-\rangle\!\rangle$	.	1	1	.	.	.	.	.	.	.	.	.	.	.
⋆	4928	$\langle\!\langle 13,2,+\rangle\!\rangle$	.	1	1	.	.	1	.	.	.	.	.	.	.	.
⋆	4928	$\langle\!\langle 13,2,-\rangle\!\rangle$	1	.	.	1	1	.	.	.	.	.	.	.	.	.
	17472	$\langle\!\langle 12,3,+\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	.	.	.
	17472	$\langle\!\langle 12,3,-\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	.	.	.
	17600	$\langle\!\langle 12,2,1\rangle\!\rangle$	2	2	.	.	1	1	1	1	.	.	.	.	.	.
⋆	40768	$\langle\!\langle 11,4,+\rangle\!\rangle$	.	1	1	.	.	1	1	1	.	1	.	.	.	.
⋆	40768	$\langle\!\langle 11,4,-\rangle\!\rangle$	1	.	.	1	1	.	1	1	1	.	.	.	.	.
	83200	$\langle\!\langle 11,3,1\rangle\!\rangle$	2	2	2	2	1	1	2	2	1	1	.	.	.	.
⋆	64064	$\langle\!\langle 10,5,+\rangle\!\rangle$	.	1	1	.	.	.	1	1	.	1	.	1	.	.
	64064	$\langle\!\langle 10,5,-\rangle\!\rangle$	1	.	.	1	.	.	1	1	1	.	1	.	.	.
	137984	$\langle\!\langle 10,3,2\rangle\!\rangle$	2	2	2	2	1	1	2	2	1	1	1	1	.	.
	64064	$\langle\!\langle 9,6,+\rangle\!\rangle$	1	1	.	.	.	.	1	1	.	.	1	1	.	.
	64064	$\langle\!\langle 9,6,-\rangle\!\rangle$	1	1	.	.	.	.	1	1	.	.	1	1	.	.
	292864	$\langle\!\langle 9,5,1\rangle\!\rangle$	2	2	2	2	.	.	2	2	1	1	1	1	1	1
	392000	$\langle\!\langle 9,4,2\rangle\!\rangle$	6	6	2	2	2	2	5	5	1	1	2	2	1	1
	81536	$\langle\!\langle 9,3,2,1,+\rangle\!\rangle$	2	2	.	.	1	1	2	2	.	.	1	1	.	.
	81536	$\langle\!\langle 9,3,2,1,-\rangle\!\rangle$	2	2	.	.	1	1	2	2	.	.	1	1	.	.
⋆	27456	$\langle\!\langle 8,7,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	1	.	.
⋆	27456	$\langle\!\langle 8,7,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	1	.	.	.
	228800	$\langle\!\langle 8,6,1\rangle\!\rangle$	2	2	1	1	.	.	1	1	.	.	1	1	1	1
	312000	$\langle\!\langle 8,4,3\rangle\!\rangle$	4	4	3	3	1	1	3	3	1	1	1	1	1	1
⋆	224224	$\langle\!\langle 8,4,2,1,+\rangle\!\rangle$	3	3	2	2	1	1	2	3	1	1	1	1	1	.
⋆	224224	$\langle\!\langle 8,4,2,1,-\rangle\!\rangle$	3	3	2	2	1	1	3	2	1	1	1	1	.	1
	246400	$\langle\!\langle 7,6,2\rangle\!\rangle$	4	4	1	1	1	1	2	2	.	.	1	1	1	1
	384384	$\langle\!\langle 7,5,3\rangle\!\rangle$	6	6	3	3	2	2	4	4	1	1	2	2	1	1
	228800	$\langle\!\langle 7,5,2,1,+\rangle\!\rangle$	3	3	2	2	1	1	3	3	1	1	1	1	1	.
⋆	228800	$\langle\!\langle 7,5,2,1,-\rangle\!\rangle$	3	3	2	2	1	1	3	3	1	1	1	1	.	1
	81536	$\langle\!\langle 6,5,4\rangle\!\rangle$	2	2	.	.	1	1	2	2	.	.	1	1	.	.
	145600	$\langle\!\langle 6,5,3,1,+\rangle\!\rangle$	2	2	1	1	1	1	3	3	1	1	1	1	.	.
	145600	$\langle\!\langle 6,5,3,1,-\rangle\!\rangle$	2	2	1	1	1	1	3	3	1	1	1	1	.	.
	64064	$\langle\!\langle 6,4,3,2,+\rangle\!\rangle$	.	.	1	1	.	.	1	1	1	1	.	.	.	.
	64064	$\langle\!\langle 6,4,3,2,-\rangle\!\rangle$	.	.	1	1	.	.	1	1	1	1	.	.	.	.
⋆	9152	$\langle\!\langle 5,4,3,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	.

11 The 5-modular spin characters of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

$\tilde{S}_{15}$ has two spin 5-blocks of defect 0, each of which covers two such blocks of $\tilde{A}_{15}$. We collect the block constants of all other spin 5-blocks of $\tilde{S}_{15}$ and $\tilde{A}_{15}$ in Table 11.1.

Table 11.1: Spin 5-blocks of non-zero defect of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
27	30	20	3	$()$	3	16	27	10
28	4	2	1	$(6,3,1)$	1	17	5	4
29	4	2	1	$(7,2,1)$	1	18	5	4

As in Section 10, we compute the 5-modular basic spin character $\sigma(15)$ of $\tilde{S}_{15}$ directly. Then $\sigma\langle\langle 15\rangle\rangle|_{3'} = \sigma(15) + \sigma(15)^a$. The 5-modular basic spin character of $\tilde{A}_{15}$ is $\lambda(15) := \sigma(15)\downarrow_{\tilde{A}_{15}}$. (Note that $\lambda(15) = \lambda\langle\langle 15,+\rangle\rangle|_{5'} = \lambda\langle\langle 15,-\rangle\rangle|_{5'}$.) By (5.5.4), $\lambda(14,1) := \lambda\langle\langle 14,1\rangle\rangle|_{5'} - \lambda(15)$ is an irreducible Brauer character of $\tilde{A}_{15}$.

11.1 Spin blocks of defect 1

The Brauer trees of B_{28}, B_{29}, b_{17}, and b_{18} are given in Appendix A as Figures A.9, A.10, A.11, and A.12, respectively.

11.2 The block b_{16}

We have a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1,\dots,10\}$ defined by Table 11.2 where $\underline{\Lambda}(b_{17}) := \lambda\langle\langle 8,6,1,+\rangle\rangle + \lambda\langle\langle 6,5,3,1\rangle\rangle$ is a projective indecomposable character of $\tilde{A}_{15}$ which can be read off the Brauer tree of b_{17} (see Λ_6). The mutual scalar products of the basic set characters are given in Table 11.3. We see that $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1,2,3\}$ and $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{8,9,10\}$.

Table 11.2: Some characters for $\mathfrak{b}_{16}$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(15)$	64	$\Lambda[\![5^2,4]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	4600000
2	$\lambda(14,1)$	768	$\Lambda[\![6,5,3,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	1336000
3	$\lambda\langle\!\langle 5,4,3,2,1,+\rangle\!\rangle\vert_{5'}$	4576	$\Lambda[\![5,4,3,2]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	3092000
4	$\lambda\langle\!\langle 13,2\rangle\!\rangle\vert_{5'}$	4928	$\Lambda[\![7,5,2,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	920000
5	$\lambda\langle\!\langle 12,3\rangle\!\rangle\vert_{5'}$	17472	$\Lambda[\![8,5,1,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	1296000
6	$\lambda(15)\otimes\lambda[9,6]$	24128	$\underline{\Lambda}(\mathfrak{b}_{17})\otimes\lambda[14,1]\ \vert_{\mathfrak{b}_{16}}$	1896000
7	$\lambda\langle\!\langle 6,5,4,+\rangle\!\rangle\vert_{5'}$	40768	$\Lambda[\![6,4,3,1,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	1844000
8	$\lambda\langle\!\langle 11,4\rangle\!\rangle\vert_{5'}$	40768	$\Sigma[\![9,4,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	728000
9	$\lambda\langle\!\langle 10,3,2,+\rangle\!\rangle\vert_{5'}$	68992	$\Lambda[\![8,4,2]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	1452000
10	$\lambda[\![6,4,3,1,+]\!]\uparrow^{\tilde{A}_{15}}$	188160	$\Lambda[\![6,3,2,1,+:2,1]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}/2$	1324000
11	$\lambda\langle\!\langle 6,4,3,2\rangle\!\rangle\vert_{5'}$	64064		
12	$\lambda\langle\!\langle 9,6\rangle\!\rangle\vert_{5'}$	64064		
13	$\lambda\langle\!\langle 8,7\rangle\!\rangle\vert_{5'}$	27456		
14	$\sigma[\![7,5,1,+]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	52480		
15	$\lambda\langle\!\langle 9,3,2,1\rangle\!\rangle\vert_{5'}$	81536		
16	$\lambda[\![7,4,3]\!]\uparrow^{\tilde{A}_{15}}\vert_{\mathfrak{b}_{16}}$	447360		
17	$\lambda\langle\!\langle 10,5\rangle\!\rangle\vert_{5'}$	64064		

Table 11.3: Mutual scalar products ($\mathfrak{b}_{16}$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
λ_1	1	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.
λ_3	.	.	1	.	.	.	.	.	.	.
λ_4	.	1	.	1	.	.	.	.	.	.
λ_5	.	1	.	1	1	.	.	.	.	.
λ_6	1	.	2	.	.	1	.	.	.	.
λ_7	.	1	.	.	.	1	1	.	.	.
λ_8	2	1	.	.	1	.	.	1	.	.
λ_9	.	.	.	.	1	.	.	.	1	.
λ_{10}	.	.	.	.	.	.	2	.	.	1

As $\Lambda_{13} = \lambda_3^* + 2\lambda_6^*$ with $\langle \lambda_{13}, \lambda_3^* \rangle = -2$, $\langle \lambda_{13}, \lambda_3^* + \lambda_6^* \rangle = -1$, and $\langle \lambda_{15}, \lambda_6^* \rangle = -1$, $\underline{\Lambda}_3 := \Lambda_3$ is indecomposable, see (2.10). The same argument holds for $\underline{\Lambda}_4 := \Lambda_4$ since $\langle \lambda_{14}, \lambda_4^* \rangle = -1$ and $\langle \lambda_{15}, \lambda_5^* \rangle = -1$, and for $\underline{\Lambda}_6 := \Lambda_6$ because $\langle \lambda_{15}, \lambda_6^* \rangle = -1$ and $\deg \lambda_7^* = -804000$. Hence we can improve λ_6 by $\lambda_{6a} := \lambda_6 - 2\underline{\lambda}_3$. The relation $\lambda_{11} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \underline{\lambda}_3 + \lambda_{6a} + \lambda_7$ leads to $\underline{\lambda}_6 := \lambda_{6a} - \underline{\lambda}_1$ (which is a Brauer atom and therefore irreducible) and $\lambda_{7a} := \lambda_7 - \underline{\lambda}_2$. As $\underline{\lambda}_6$ corresponds to $\underline{\Lambda}_6$ and $\langle \lambda_{7a}, \underline{\Lambda}_6 \rangle = 1$, we get $\underline{\lambda}_7 := \lambda_{7a} - \underline{\lambda}_6$ which is irreducible too. Moreover, we get $\lambda_{8a} := \lambda_8 - \underline{\lambda}_2$ from $\lambda_{12} = -\underline{\lambda}_2 + 2\underline{\lambda}_3 + \underline{\lambda}_6 + \lambda_8$. We substitute $\{\lambda_6, \lambda_7, \lambda_8\}$ with $\{\underline{\lambda}_6, \underline{\lambda}_7, \lambda_{8a}\}$ in the basic set of Brauer characters.

Table 11.4: Mutual scalar products (b_{16})

$\langle \cdot,\cdot \rangle$	Λ_1	Λ_2	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	Λ_5	$\underline{\Lambda}_6$	Λ_7	$\underline{\Lambda}_8$	$\underline{\Lambda}_9$	$\underline{\Lambda}_{10}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.	.	.	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.	.	.	.	.	.
λ_4	.	1	.	1	.	.	.	.	.	.
λ_5	.	1	.	1	1	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	.	1	.	.	.	.
$\underline{\lambda}_7$	.	.	.	.	.	.	1	.	.	.
λ_{8a}	2	.	.	.	1	.	.	1	.	.
λ_9	.	.	.	.	1	.	.	.	1	.
λ_{10}	.	.	.	.	.	.	2	.	.	1

The mutual scalar products in Table 11.4 and the relation $\lambda_{13} = -\lambda_4 + \lambda_5 + \underline{\lambda}_6$ show that we may improve λ_5 by $\lambda_{5a} := \lambda_5 - \lambda_4 + \underline{\lambda}_2$. Then $\lambda_{14} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \lambda_{5a} + \lambda_{8a}$ gives $\underline{\lambda}_5 := \lambda_{5a} - \underline{\lambda}_2$ (which is irreducible), and $\lambda_{8b} := \lambda_{8a} - \underline{\lambda}_1$. Consider the relations

$$\begin{aligned}
\lambda_{15} &= -\underline{\lambda}_5 + \underline{\lambda}_7 + \lambda_9, \\
\lambda_{16} &= -2\underline{\lambda}_2 + 2\lambda_4 - 5\underline{\lambda}_7 + 3\lambda_{10}, \\
\lambda_{17} &= 2\underline{\lambda}_1 + 2\underline{\lambda}_3 - 2\underline{\lambda}_5 + 2\lambda_{8b}.
\end{aligned}$$

The first one shows that $\underline{\lambda}_5$ is contained in λ_9. From the second one we can deduce that $\underline{\lambda}_2$ is contained in λ_4, and $\underline{\lambda}_7$ must be contained twice in λ_{10}, see Table 11.4. The third relation implies that $\underline{\lambda}_5$ may be subtracted from λ_{8b}. Then $\underline{\lambda}_9 := \lambda_9 - \underline{\lambda}_5$, $\underline{\lambda}_4 := \lambda_4 - \underline{\lambda}_2$, and $\underline{\lambda}_{10} := \lambda_{10} - 2\underline{\lambda}_7$ are Brauer atoms, hence irreducible. Only for $\lambda_{8c} := \lambda_{8b} - \underline{\lambda}_5$ it is not clear whether or not $\underline{\lambda}_1$ can be subtracted one more time.

11.3 The block B_{27}

We start with basic sets $\{\sigma_1, \ldots, \sigma_{20}\}$ and $\{\Sigma_1, \ldots, \Sigma_{20}\}$ for B_{27} defined by Table 11.5. Set $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$. Then $\underline{\sigma}_3 := \sigma_3 - \underline{\sigma}_2$ and $\underline{\sigma}_4 := \sigma_4 - \underline{\sigma}_1$ are irreducible, see (5.5.4) and Table 11.6. Also, $\underline{\Sigma}_i := \Sigma_i$ is indecomposable for $i \in \{13, 14, 17, 18, 19, 20\}$. As $\underline{\lambda}_3(b_{16})\uparrow^{\tilde{S}_{15}}= \sigma_7 = \Sigma_7^* + \Sigma_8^*$ and $\underline{\lambda}_7(b_{16})\uparrow^{\tilde{S}_{15}}= \sigma_{14} = \Sigma_{15}^* + \Sigma_{16}^*$, we see that $\underline{\sigma}_7 := \Sigma_7^*$, $\underline{\sigma}_{14} := \Sigma_8^*$, $\underline{\sigma}_{15} := \Sigma_{15}^*$, and $\underline{\sigma}_{16} := \Sigma_{16}^*$ are irreducible Brauer characters. Similarly, $\underline{\Lambda}_3(b_{16})\uparrow^{\tilde{S}_{15}}= \Sigma_7 + \Sigma_8$, $\underline{\Lambda}_4(b_{16})\uparrow^{\tilde{S}_{15}}= \Sigma_5 + \Sigma_6$, and $\underline{\Lambda}_5(b_{16})\uparrow^{\tilde{S}_{15}}= \Sigma_9 + \Sigma_{10}$, show that $\underline{\Sigma}_i := \Sigma_i$, $i \in \{5, \ldots, 10\}$, are indecomposable. Then $\underline{\Sigma}_7$ and $\underline{\Sigma}_8$ correspond to $\underline{\sigma}_7$ and $\underline{\sigma}_{14}$, respectively. Hence we can improve σ_i by $\sigma_{ia} := \sigma_i - \underline{\sigma}_7 - \underline{\sigma}_{14}$ for $i \in \{10, 11, 19, 20\}$. Then it follows from $\underline{\lambda}_6(b_{16})\uparrow^{\tilde{S}_{15}}= -\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_{10a} + \sigma_{11a}$ and Table 11.7 that $\underline{\sigma}_{10} := \sigma_{10a} - \underline{\sigma}_2$ and $\underline{\sigma}_{11} := \sigma_{11a} - \underline{\sigma}_1$ are irreducible. Further relations

$$\begin{aligned}
\sigma_{21} &= -\underline{\sigma}_2 - \underline{\sigma}_3 - \sigma_5 + \sigma_8 + \sigma_{12},\\
\sigma_{22} &= -\underline{\sigma}_1 - \underline{\sigma}_4 - \sigma_6 + \sigma_9 + \sigma_{13},\\
\sigma_{23} &= -\underline{\sigma}_{15} + \sigma_8 + \sigma_9 + \sigma_{18},\\
\sigma_{24} &= -\underline{\sigma}_{16} + \sigma_8 + \sigma_9 + \sigma_{17},\\
\sigma_{25} &= \underline{\sigma}_1 + \underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_6,\\
\sigma_{26} &= \underline{\sigma}_2 - \underline{\sigma}_3 + \underline{\sigma}_4 + \sigma_5
\end{aligned}$$

lead to the Brauer characters $\sigma_{12a} := \sigma_{12} - \underline{\sigma}_2$, $\sigma_{13a} := \sigma_{13} - \underline{\sigma}_1$, $\sigma_{18a} := \sigma_{18} - \underline{\sigma}_{15}$, $\sigma_{17a} := \sigma_{17} - \underline{\sigma}_{16}$, $\sigma_{6a} := \sigma_6 - \underline{\sigma}_4$, and $\sigma_{5a} := \sigma_5 - \underline{\sigma}_3$, respectively.
Since $\sigma_{27} = \underline{\lambda}_{14} - \underline{\lambda}_{10} - \underline{\sigma}_{11} + \sigma_{18a} + \sigma_{19a}$ and $\sigma_{28} = \underline{\lambda}_7 - \underline{\lambda}_{10} - \underline{\sigma}_{11} + \sigma_{17a} + \sigma_{20a}$, we can improve σ_{ia} by $\sigma_{ib} := \sigma_{ia} - \underline{\sigma}_{10} - \underline{\sigma}_{11}$ for $i \in \{19, 20\}$. The mutual scalar products, see Table 11.8, and the relation $\Sigma_{21} = \Sigma_3 + \Sigma_4 - 2\underline{\Sigma}_5 - 2\underline{\Sigma}_6$ give $\Sigma_{3a} := \Sigma_3 - 2\underline{\Sigma}_5$ and $\Sigma_{4a} := \Sigma_4 - 2\underline{\Sigma}_6$ instead of Σ_3 and Σ_4, respectively. In particular, $\underline{\sigma}_5 := \sigma_{5a}$ and $\underline{\sigma}_6 := \sigma_{6a}$ are irreducible.

Table 11.5: Some characters for B_{27}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(15)$	64	$\Sigma[\![5^2,4,+]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	4600000
2	$\sigma(15)^a$	64	$\Sigma[\![5^2,4,-]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	4600000
3	$\sigma\langle\!\langle 14,1,+\rangle\!\rangle\vert_{5'}$	832	$\sigma(15)\otimes\Sigma[3^4,2,1]\mid_{B_{27}}$	3176000
4	$\sigma\langle\!\langle 14,1,-\rangle\!\rangle\vert_{5'}$	832	$\sigma(15)\otimes\Sigma[12,2,1]\mid_{B_{27}}$	3176000
5	$\sigma\langle\!\langle 13,2,-\rangle\!\rangle\vert_{5'}$	4928	$\sigma(15)\otimes\Sigma[3^4,1^3]\mid_{B_{27}}$	920000
6	$\sigma\langle\!\langle 13,2,+\rangle\!\rangle\vert_{5'}$	4928	$\sigma(15)\otimes\Sigma[12,3]\mid_{B_{27}}$	920000
7	$\sigma\langle\!\langle 5,4,3,2,1\rangle\!\rangle\vert_{5'}$	9152	$\Sigma[\![5,4,3,2,+]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	3092000
8	$\sigma\langle\!\langle 12,3,-\rangle\!\rangle\vert_{5'}$	17472	$\Sigma[\![5,4,3,2,-]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	3092000
9	$\sigma\langle\!\langle 12,3,+\rangle\!\rangle\vert_{5'}$	17472	$\sigma(15)\otimes\Sigma[11,4]\mid_{B_{27}}$	1296000
10	$\sigma(15)\otimes\sigma[9,6]$	24128	$\sigma(15)\otimes\Sigma[3^3,2,1^4]\mid_{B_{27}}$	1296000
11	$\sigma(15)\otimes\sigma[5^3]$	24128	$\sigma(15)\otimes\Sigma[6^2,2,1]\mid_{B_{27}}$	4544000
12	$\sigma\langle\!\langle 11,4,-\rangle\!\rangle\vert_{5'}$	40768	$\sigma(15)\otimes\Sigma[7,6,2]\mid_{B_{27}}$	4544000
13	$\sigma\langle\!\langle 11,4,+\rangle\!\rangle\vert_{5'}$	40768	$\sigma(15)\otimes\Sigma[9,5,1]$	728000
14	$\underline{\lambda}_7(b_{16})\uparrow^{\tilde{S}15}$	50176	$\sigma(15)\otimes\Sigma[3,2^4,1^4]$	728000
15	$\sigma\langle\!\langle 6,4,3,2,-\rangle\!\rangle\vert_{5'}$	64064	$\Sigma[\![6,4,3:2,-]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	4620000
16	$\sigma[\![5,4,3,2,-]\!]\uparrow^{\tilde{S}15}$	68640	$\Sigma[\![6,4,3:2,+]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	4620000
17	$\sigma\langle\!\langle 9,3,2,1,+\rangle\!\rangle\vert_{5'}$	81536	$\Sigma[\![8,4,2,+]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	1452000
18	$\sigma\langle\!\langle 9,3,2,1,-\rangle\!\rangle\vert_{5'}$	81536	$\Sigma[\![8,4,2,-]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	1452000
19	$\sigma\langle\!\langle 7,4,3,1,+\rangle\!\rangle\vert_{5'}$	202176	$\Sigma[\![7,4,3,+]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	1324000
20	$\sigma\langle\!\langle 7,4,3,1,-\rangle\!\rangle\vert_{5'}$	202176	$\Sigma[\![7,4,3,-]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	1324000
21	$\sigma(15)\otimes\sigma[11,4]$	52480	$\Sigma[\![6,5,3]\!]\uparrow^{\tilde{S}_{15}}\mid_{B_{27}}$	2672000
22	$\sigma(15)^a\otimes\sigma[11,4]$	52480		
23	$\sigma[\![5,4,1:3,2,+]\!]\uparrow^{\tilde{S}15}\mid_{B_{27}}$	91392		
24	$\sigma[\![5,4,1:3,2,-]\!]\uparrow^{\tilde{S}15}\mid_{B_{27}}$	91392		
25	$\sigma(15)\otimes\sigma[13,1^2]$	4992		
26	$\sigma(15)\otimes\sigma[4,3^3,2]$	4992		
27	$\sigma\langle\!\langle 8,4,2,1,-\rangle\!\rangle\vert_{5'}$	224224		
28	$\sigma\langle\!\langle 8,4,2,1,+\rangle\!\rangle\vert_{5'}$	224224		

Table 11.6: Mutual scalar products (B_{27})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}	Σ_{19}	Σ_{20}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_4	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_5	.	.	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_6	.	.	2	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_7	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
σ_8	.	.	1	2	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
σ_9	.	.	2	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
σ_{10}	.	1	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.
σ_{11}	1	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.
σ_{12}	1	1	1	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.
σ_{13}	1	1	.	1	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.
σ_{14}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
σ_{15}	.	.	.	.	.	.	1	1	.	.	1	1	.	.	1	.	.	.	.	.
σ_{16}	.	.	.	.	.	.	1	2	.	.	1	1	.	.	.	1	.	.	.	.
σ_{17}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.
σ_{18}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.
σ_{19}	1	1	.	.	.	.	1	1	.	.	2	2	.	.	1	1	.	.	1	.
σ_{20}	1	1	.	.	.	.	1	1	.	.	2	2	.	.	1	1	.	.	.	1

Table 11.7: Mutual scalar products (B_{27})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	Σ_{15}	Σ_{16}	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	Σ_{11}	Σ_{12}
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_7$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_{14}$	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_{15}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_{16}$	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	.	.	.	.	.	1	.	.	.	.	.	.
σ_5	.	.	.	.	.	.	1	2	1	.	.	.	.	.
σ_6	.	.	.	.	.	.	2	1	.	1	.	.	.	.
σ_8	.	.	.	.	.	.	1	2	1	.	1	.	.	.
σ_9	.	.	.	.	.	.	2	1	.	1	.	1	.	.
σ_{10}	.	1	1	1	.	.	.	.	.	.	.	.	1	.
σ_{11}	1	.	1	1	.	.	.	.	.	.	.	.	.	1
σ_{12}	1	1	.	.	.	.	1	.	.	.	1	.	.	.
σ_{13}	1	1	.	.	.	.	.	1	.	.	.	1	.	.
σ_{17}	.	.	.	.	1	1	.	.	.	.	.	.	.	.
σ_{18}	.	.	.	.	1	1	.	.	.	.	.	.	.	.
σ_{19}	1	1	1	1	1	1	.	.	.	.	.	.	2	2
σ_{20}	1	1	1	1	1	1	.	.	.	.	.	.	2	2

Table 11.8: Mutual scalar products (B_{27})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	Σ_{15}	Σ_{16}	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	Σ_{11}	Σ_{12}
σ_{5a}	.	.	.	.	.	.	.	2	1	.	.	.	.	.
σ_{6a}	.	.	.	.	.	.	2	.	.	1	.	.	.	.
σ_8	.	.	.	.	.	.	1	2	1	.	1	.	.	.
σ_9	.	.	.	.	.	.	2	1	.	1	.	1	.	.
$\underline{\sigma}_{10}$	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$\underline{\sigma}_{11}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1
σ_{12a}	1	.	.	.	.	.	1	.	.	.	1	.	.	.
σ_{13a}	.	1	.	.	.	.	.	1	.	.	.	1	.	.
σ_{17a}	.	.	.	.	1	.	.	.	.	.	.	.	.	.
σ_{18a}	.	.	.	.	.	1	.	.	.	.	.	.	.	.
σ_{19a}	1	1	.	.	1	1	.	.	.	.	.	.	1	1
σ_{20a}	1	1	.	.	1	1	.	.	.	.	.	.	1	1

Now

$$\sigma_{22} = -2\underline{\sigma}_3 - \underline{\sigma}_5 + \sigma_8 + \sigma_{12a},$$
$$\sigma_{21} = -2\underline{\sigma}_4 - \underline{\sigma}_6 + \sigma_9 + \sigma_{13a},$$
$$\underline{\Lambda}_{10}(b_{16})\uparrow^{\tilde{S}_{15}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_{15} - \underline{\sigma}_{16} + \sigma_{19a} + \sigma_{20a}$$

imply that σ_8 and σ_9 contain the irreducible Brauer characters $\underline{\sigma}_8 := \sigma_8 - \underline{\sigma}_3 - \underline{\sigma}_5$ and $\underline{\sigma}_9 := \sigma_9 - \underline{\sigma}_4 - \underline{\sigma}_6$, respectively. We improve σ_{12a}, σ_{13a}, σ_{19a} and σ_{20a} by $\sigma_{12b} := \sigma_{12a} - \underline{\sigma}_3$, $\sigma_{13b} := \sigma_{13a} - \underline{\sigma}_4$, $\sigma_{19b} := \sigma_{19a} - \underline{\sigma}_1 - \underline{\sigma}_2$, and $\sigma_{20b} := \sigma_{20a} - \underline{\sigma}_1 - \underline{\sigma}_2$, respectively. Similarly, $\underline{\Lambda}_6(b_{16})\uparrow^{\tilde{S}_{15}} = \Sigma_{11} + \Sigma_{12} - 2\underline{\Sigma}_{19} - 2\underline{\Sigma}_{20}$ leads to projective indecomposable characters $\underline{\Sigma}_i := \Sigma_i - \underline{\Sigma}_{19} - \underline{\Sigma}_{20}$, $i \in \{11, 12\}$. Now $\underline{\sigma}_8$ corresponds to $\underline{\Sigma}_9$, and $\underline{\sigma}_9$ corresponds to $\underline{\Sigma}_{10}$. Hence we get $\sigma_{12c} := \sigma_{12b} - \underline{\sigma}_8$ and $\sigma_{13c} := \sigma_{13b} - \underline{\sigma}_9$, see Table 11.9.

Table 11.9: Mutual scalar products (B_{27})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	Σ_{15}	Σ_{16}	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_8$	.	.	.	.	.	.	.	.	.	.	1	.
$\underline{\sigma}_9$	.	.	.	.	.	.	.	.	.	1	.	1
σ_{12b}	1	.	.	.	.	.	.	.	.	.	1	.
σ_{13b}	.	1	.	.	.	.	.	.	.	.	.	1
σ_{17a}	.	.	.	.	1	.	.	.	.	.	.	.
σ_{18a}	.	.	.	.	.	1	.	.	.	.	.	.
σ_{19b}	.	.	.	.	1	1	.	.	.	.	.	.
σ_{20b}	.	.	.	.	1	1	.	.	.	.	.	.

Then $\underline{\Lambda}_7(b_{16})\uparrow^{\tilde{S}_{15}} = \Sigma_{15} + \Sigma_{16} - \underline{\Sigma}_{17} - \underline{\Sigma}_{18} - \underline{\Sigma}_{19} - \underline{\Sigma}_{20}$ implies that $\underline{\Sigma}_{17}$ is contained in Σ_{15}, and $\underline{\Sigma}_{18}$ is contained in Σ_{16}. In particular, $\underline{\sigma}_{17} := \sigma_{17a}$ and $\underline{\sigma}_{18} := \sigma_{18a}$ are irreducible. Decomposing

$$\underline{\sigma}_7 \otimes \sigma[13, 1^2] = \underline{\sigma}_1 + \underline{\sigma}_2 + 2\underline{\sigma}_7 + 2\underline{\sigma}_{14} - \underline{\sigma}_{15} + \underline{\sigma}_{16} + \underline{\sigma}_8 + \underline{\sigma}_9 + \sigma_{19b}$$

shows that $\underline{\sigma}_{15}$ is contained in σ_{19b}. Thus $\underline{\sigma}_{16} = \underline{\sigma}_{15}^a$ is contained in $\sigma_{20b} = \sigma_{19b}^a$. It follows from the decomposition of $\underline{\Lambda}_7(b_{16})\uparrow^{\tilde{S}_{15}}$ (see above) that $\underline{\Sigma}_{19}$ is contained in Σ_{16}, and $\underline{\Sigma}_{20}$ is contained in Σ_{15}. In particular, $\underline{\sigma}_{19} := \sigma_{19b} - \underline{\sigma}_{15}$ and $\underline{\sigma}_{20} := \sigma_{20b} - \underline{\sigma}_{16}$ are irreducible. At this point there is no more information from character relations. Thus we are going to determine the degree (which is either 27392 or 27328) of the irreducible

Brauer character $\underline{\sigma}_{12}$ that is contained in $\sigma_{12\mathrm{b}}$ by means of condensation. Let $64\mathrm{a}^-$ denote a basic spin module of $\tilde{S}_{15}$ over $F := \mathrm{GF}(25)$ that affords $\underline{\sigma}_1$. Then $64\mathrm{a}^- \otimes 64\mathrm{a}^-$ has a constituent $715\mathrm{a}^+$ that affords $\sigma[6, 3, 2^3]$. As

$$\underline{\sigma}_1 \otimes \sigma[6, 3, 2^3] = 2\underline{\sigma}_2 + 2\underline{\sigma}_4 + \underline{\sigma}_5 + \underline{\sigma}_9 + \sigma_{12\mathrm{b}},$$

the tensor product $V := 64\mathrm{a}^- \otimes 715\mathrm{a}^+$ has a constituent with Brauer character $\underline{\sigma}_{12}$. We use the same condensation subgroup $K \cong 7^2$ with faithful trivial idempotent $e := e_K$ and the same generators for the condensation algebra $\mathcal{C} \leqslant eF\tilde{S}_{15}e$ as in Section 10. The condensed module Ve of dimension 964 has exactly five $\mathcal{C}$-constituents that correspond to $\underline{\sigma}_2$, $\underline{\sigma}_4$, $\underline{\sigma}_5$, $\underline{\sigma}_9$, and $\sigma_{12\mathrm{b}}$ with multiplicities 2, 2, 1, 1, and 1, respectively. Thus $\sigma_{12\mathrm{b}}$ does not contain $\underline{\sigma}_1$. Therefore $\underline{\sigma}_{12} = \sigma_{12\mathrm{b}}$, $\underline{\sigma}_{13} := \sigma_{13\mathrm{b}}$, and $\underline{\lambda}_8 := \lambda_{8\mathrm{c}}$ are irreducible, and we are done. The decomposition matrices of b_{16} and B_{27} are Tables 11.10 and 11.11, respectively.

Table 11.10: 5-modular decomposition matrix of block $\mathfrak{b}_{16}$ of $\tilde{A}_{15}$ (defect 3, bar core $()$, weight 3, content $(6,6,3)$)

			$[\![5,4,3,2,1]\!]$ 4576	$[\![5^2,3,2]\!]$ 14912	$[\![5^2,4,1]\!]$ 64	$[\![6,4,3,2]\!]$ 25088	$[\![6,5,4]\!]$ 768	$[\![7,4,3,1]\!]$ 137984	$[\![7,5,3]\!]$ 4160	$[\![8,4,2,1]\!]$ 56448	$[\![8,5,2]\!]$ 12544	$[\![9,5,1]\!]$ 27392
$\star$	64	$\langle\!\langle 15,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
	64	$\langle\!\langle 15,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
$\star$	832	$\langle\!\langle 14,1\rangle\!\rangle$	.	.	1	.	1	.	.	.	.	.
$\star$	4928	$\langle\!\langle 13,2\rangle\!\rangle$	.	.	.	.	1	.	1	.	.	.
$\star$	17472	$\langle\!\langle 12,3\rangle\!\rangle$	.	.	.	.	1	.	1	.	1	.
$\star$	40768	$\langle\!\langle 11,4\rangle\!\rangle$	.	.	1	.	1	.	.	.	1	1
	64064	$\langle\!\langle 10,5\rangle\!\rangle$	2	.	2	.	.	.	.	.	.	2
	101088	$\langle\!\langle 10,4,1,+\rangle\!\rangle$	1	.	2	.	.	.	.	1	1	1
	101088	$\langle\!\langle 10,4,1,-\rangle\!\rangle$	1	.	2	.	.	.	.	1	1	1
$\star$	68992	$\langle\!\langle 10,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	1	.
	68992	$\langle\!\langle 10,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	1	1	.
	64064	$\langle\!\langle 9,6\rangle\!\rangle$	2	1	1	.	.	.	.	.	1	1
	146432	$\langle\!\langle 9,5,1,+\rangle\!\rangle$	2	1	2	1	1	.	.	1	1	1
	146432	$\langle\!\langle 9,5,1,-\rangle\!\rangle$	2	1	2	1	1	.	.	1	1	1
	81536	$\langle\!\langle 9,3,2,1\rangle\!\rangle$	.	.	.	1	.	.	.	1	.	.
$\star$	27456	$\langle\!\langle 8,7\rangle\!\rangle$	.	1	.	.	.	.	.	.	1	.
	256608	$\langle\!\langle 8,5,2,+\rangle\!\rangle$	1	1	2	1	1	1	1	1	1	.
	256608	$\langle\!\langle 8,5,2,-\rangle\!\rangle$	1	1	2	1	1	1	1	1	1	.
	224224	$\langle\!\langle 8,4,2,1\rangle\!\rangle$	1	.	2	1	.	1	.	1	.	.
$\star$	192192	$\langle\!\langle 7,5,3,+\rangle\!\rangle$	2	1	2	1	1	1	1	.	.	.
	192192	$\langle\!\langle 7,5,3,-\rangle\!\rangle$	2	1	2	1	1	1	1	.	.	.
	202176	$\langle\!\langle 7,4,3,1\rangle\!\rangle$	2	2	2	1	.	1	.	.	.	.
$\star$	40768	$\langle\!\langle 6,5,4,+\rangle\!\rangle$	.	1	.	1	1	.	.	.	.	.
	40768	$\langle\!\langle 6,5,4,-\rangle\!\rangle$	.	1	.	1	1	.	.	.	.	.
	64064	$\langle\!\langle 6,4,3,2\rangle\!\rangle$	2	2	.	1	.	.	.	.	.	.
$\star$	4576	$\langle\!\langle 5,4,3,2,1,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.
	4576	$\langle\!\langle 5,4,3,2,1,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.

Table 11.11: 5-modular decomposition matrix of block B_{27} of $\tilde{S}_{15}$
(defect 3, bar core $()$, weight 3, content $(6,6,3)$)

			$[\![5,4,3,2,1,+]\!]$ 4576	$[\![5,4,3,2,1,-]\!]$ 4576	$[\![5^2,3,2,+]\!]$ 14912	$[\![5^2,3,2,-]\!]$ 14912	$[\![5^2,4,1,+]\!]$ 64	$[\![5^2,4,1,-]\!]$ 64	$[\![6,4,3,2,+]\!]$ 25088	$[\![6,4,3,2,-]\!]$ 25088	$[\![6,5,4,+]\!]$ 768	$[\![6,5,4,-]\!]$ 768	$[\![7,4,3,1,+]\!]$ 137984	$[\![7,4,3,1,-]\!]$ 137984	$[\![7,5,3,+]\!]$ 4160	$[\![7,5,3,-]\!]$ 4160	$[\![8,4,2,1,+]\!]$ 56448	$[\![8,4,2,1,-]\!]$ 56448	$[\![8,5,2,+]\!]$ 12544	$[\![8,5,2,-]\!]$ 12544	$[\![9,5,1,+]\!]$ 27392	$[\![9,5,1,-]\!]$ 27392
$\star$	128	$\langle\!\langle 15\rangle\!\rangle$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	832	$\langle\!\langle 14,1,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\star$	832	$\langle\!\langle 14,1,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\star$	4928	$\langle\!\langle 13,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
$\star$	4928	$\langle\!\langle 13,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.
$\star$	17472	$\langle\!\langle 12,3,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
$\star$	17472	$\langle\!\langle 12,3,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.
$\star$	40768	$\langle\!\langle 11,4,+\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1
$\star$	40768	$\langle\!\langle 11,4,-\rangle\!\rangle$	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	1	.	1	.
	64064	$\langle\!\langle 10,5,+\rangle\!\rangle$	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1
	64064	$\langle\!\langle 10,5,-\rangle\!\rangle$	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1
	202176	$\langle\!\langle 10,4,1\rangle\!\rangle$	1	1	.	.	2	2	.	.	.	.	.	.	.	.	1	1	1	1	1	1
	137984	$\langle\!\langle 10,3,2\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	.	.
$\star$	64064	$\langle\!\langle 9,6,+\rangle\!\rangle$	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.
	64064	$\langle\!\langle 9,6,-\rangle\!\rangle$	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1
	292864	$\langle\!\langle 9,5,1\rangle\!\rangle$	2	2	1	1	2	2	1	1	1	1	.	.	.	.	1	1	1	1	1	1
$\star$	81536	$\langle\!\langle 9,3,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.
$\star$	81536	$\langle\!\langle 9,3,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.
$\star$	27456	$\langle\!\langle 8,7,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$\star$	27456	$\langle\!\langle 8,7,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
	513216	$\langle\!\langle 8,5,2\rangle\!\rangle$	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	.	.
$\star$	224224	$\langle\!\langle 8,4,2,1,+\rangle\!\rangle$	.	1	.	.	1	1	.	1	.	.	1	.	.	.	1	.	.	.	.	.
	224224	$\langle\!\langle 8,4,2,1,-\rangle\!\rangle$	1	.	.	.	1	1	1	.	.	.	.	1	.	.	.	1	.	.	.	.
	384384	$\langle\!\langle 7,5,3\rangle\!\rangle$	2	2	1	1	2	2	1	1	1	1	1	1	1	1	.	.	.	.	.	.
$\star$	202176	$\langle\!\langle 7,4,3,1,+\rangle\!\rangle$	1	1	1	1	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.
$\star$	202176	$\langle\!\langle 7,4,3,1,-\rangle\!\rangle$	1	1	1	1	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.
	81536	$\langle\!\langle 6,5,4\rangle\!\rangle$	.	.	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.
$\star$	64064	$\langle\!\langle 6,4,3,2,+\rangle\!\rangle$	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	64064	$\langle\!\langle 6,4,3,2,-\rangle\!\rangle$	1	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	9152	$\langle\!\langle 5,4,3,2,1\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

12 The 7-modular spin characters of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

Both $\tilde{S}_{15}$ and $\tilde{A}_{15}$ have nine faithful 7-blocks of defect 0. The block constants of all other faithful 7-blocks of $\tilde{S}_{15}$ and $\tilde{A}_{15}$ are given in Table 12.1.

Table 12.1: Spin 7-blocks of non-zero defect of $\tilde{S}_{15}$ and $\tilde{A}_{15}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
53	17	9	2	(1)	2	28	22	18
54	7	6	1	$(6,2)$	1	29	5	3
55	7	6	1	$(5,3)$	1	30	5	3

The 7-modular basic spin character of $\tilde{S}_{15}$ is $\sigma(15) := \sigma\langle\langle 15\rangle\rangle|_{7'}$. Thus $\lambda(15) := \lambda\langle\langle 15,+\rangle\rangle|_{7'}$ and $\lambda(15)^c = \lambda\langle\langle 15,-\rangle\rangle|_{7'}$ are the 7-modular basic spin characters of $\tilde{A}_{15}$. Another irreducible Brauer character of $\tilde{S}_{15}$ is $\sigma(14,1) := \sigma\langle\langle 14,1,+\rangle\rangle|_{7'} - \sigma(15)$.

12.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{54}, B_{54}, b_{29}, and b_{30} are given in Appendix A as Figures A.13, A.14, A.15, and A.16, respectively.

12.2 The block B_{53}

Basic sets of Brauer characters σ_i and projective characters Σ_i for $i \in \{1,\dots,9\}$ are defined by Table 12.2. Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1,2\}$. Clearly, $\underline{\sigma}_i := \sigma_i$ is irreducible with corresponding projective indecomposable character $\underline{\Sigma}_i := \Sigma_i$ for $i \in \{4,\dots,8\}$, see Table 12.3. We also see that $\underline{\sigma}_3 := \sigma_3$ is irreducible and $\underline{\Sigma}_9$ is indecomposable. As $\langle\sigma_{10},\underline{\Sigma}_4\rangle = \langle\sigma_{10},\underline{\Sigma}_6\rangle = 4$ and $\langle\sigma_{10},\underline{\Sigma}_7\rangle = 3$, we get $\underline{\sigma}_9 := \sigma_{10} - 4\underline{\sigma}_4 - 4\underline{\sigma}_6 - 3\underline{\sigma}_7$ which is equal to $\underline{\Sigma}_9^*$, thus irreducible with $\langle\underline{\sigma}_9,\underline{\Sigma}_9\rangle = 1$. It follows that $\underline{\Sigma}_3 := \Sigma_3$ is the projective indecomposable character corresponding to $\underline{\sigma}_3$. Then $\langle\sigma_{11},\underline{\Sigma}_3\rangle = \langle\sigma_{11},\underline{\Sigma}_9\rangle = 2$ and we obtain the irreducible Brauer character $\underline{\sigma}_5 := \sigma_{11} - 2\underline{\sigma}_3 - 2\underline{\sigma}_9 = \underline{\Sigma}_5^*$ which completes $\mathrm{IBr}(B_{53})$. The decomposition matrix of B_{53} is Table 12.6.

Table 12.2: Some characters for B_{53}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(15)$	128	$\Sigma[\![7,6:2,-]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	514304
2	$\sigma(14,1)$	704	$\Sigma[\![8,6]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	595840
3	$\sigma\langle\!\langle 5,4,3,2,1\rangle\!\rangle\|_{7'}$	9152	$\Sigma[\![5,4,3:2,1]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	2355136
4	$\sigma[\![9,5]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	16896	$\Sigma[\![9,5]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	595840
5	$\sigma\langle\!\langle 8,7,-\rangle\!\rangle\|_{7'}$	27456	$\Sigma[\![6,5,2:2,+]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	1483328
6	$\sigma[\![10,4]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	66304	$\Sigma[\![10,3,1,+]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	285376
7	$\sigma[\![7,4,3]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	118976	$\sigma(15)\otimes\Sigma[8,4,3]\|_{B_{53}}/2$	1320256
8	$\sigma[\![7,5,2]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	156288	$\Sigma[\![8,5,1,+]\!]\uparrow^{\tilde{S}_{15}}$	1034880
9	$\sigma\langle\!\langle 7,4,3,1,-\rangle\!\rangle\|_{7'}$	202176	$\Sigma[\![7,4,2,1,-]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	1687168
10	$\sigma[\![7,1:4,3]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	882752		
11	$\sigma[\![7,4,2,1,+]\!]\uparrow^{\tilde{S}_{15}}\|_{B_{53}}$	430976		

Table 12.3: Mutual scalar products (B_{53})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9
σ_1	1	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.
σ_4	.	.	.	1	.	.	.	.	.
σ_5	1	1	.	.	1	.	.	.	.
σ_6	.	.	.	.	.	1	.	.	.
σ_7	.	.	.	.	.	.	1	.	.
σ_8	.	.	.	.	.	.	.	1	.
σ_9	.	.	1	.	.	.	.	.	1

12.3 The block b_{28}

We have basic sets $\{\lambda_1, \dots, \lambda_{18}\}$ and $\{\Lambda_1, \dots, \Lambda_{18}\}$, see Tables 12.4 and 12.5.

Table 12.4: Some characters for b_{28}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\underline{\lambda}(15)$	64	$\underline{\lambda}(15) \otimes \Lambda[13, 1^2] \mid_{b_{30}}$	257152
2	$\underline{\lambda}(15)^c$	64	$\underline{\lambda}(15)^c \otimes \Lambda[13, 1^2] \mid_{b_{30}}$	257152
3	$\lambda[\![8, 6, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	352	$\Lambda[\![8, 6, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	297920
4	$\lambda[\![8, 6, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	352	$\Lambda[\![8, 4, + : 3, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1773408
5	$\lambda\langle\!\langle 5, 4, 3, 2, 1, +\rangle\!\rangle\mid_{7'}$	4576	$\Lambda[\![8, 6, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	297920
6	$\lambda\langle\!\langle 5, 4, 3, 2, 1, -\rangle\!\rangle\mid_{7'}$	4576	$\Sigma[\![8, 5, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	815360
7	$\lambda[\![9, 5, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	8448	$\Lambda[\![8, 4, + : 3, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1773408
8	$\lambda[\![9, 5, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	8448	$\Lambda[\![5, 4, 3, 2, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1177568
9	$\underline{\sigma}_5(B_{53})\downarrow_{\tilde{A}_{15}}$	26624	$\Lambda[\![5, 4, 3, 2, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1177568
10	$\lambda[\![10, 4, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	33152	$\Lambda[\![9, 5, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	297920
11	$\lambda[\![10, 4, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	33152	$\Lambda[\![9, 5, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	297920
12	$\lambda[\![7, 4, 3, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	59488	$\Lambda[\![7, 5, 2, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1259104
13	$\lambda[\![7, 4, 3, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	59488	$\Lambda[\![7, 5, 2, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1259104
14	$\lambda[\![7, 5, 2, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	78144	$\Lambda[\![10, 4, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	142688
15	$\lambda[\![7, 5, 2, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	78144	$\Lambda[\![10, 4, -]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	142688
16	$\lambda\langle\!\langle 8, 4, 3, -\rangle\!\rangle\mid_{7'}$	156000	$\Lambda[\![7, 4, 3, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1503712
17	$\lambda\langle\!\langle 8, 4, 3, +\rangle\!\rangle\mid_{7'}$	156000	$\underline{\Lambda}(15) \otimes \Lambda[6^2, 1^3] \mid_{b_{30}}$	1034880
18	$\lambda\langle\!\langle 8, 5, 2, -\rangle\!\rangle\mid_{7'}$	256608	$\Lambda[\![6, 4, 3, 1, +]\!]\uparrow^{\tilde{A}_{15}} \mid_{b_{30}}$	1687168

First $\underline{\sigma}_i(B_{53})\downarrow_{\tilde{A}_{15}} = \sigma_{2i-1} + \sigma_{2i}$ for $i \in \{1, 2, 3, 4, 6, 7, 8\}$ implies that $\underline{\sigma}_j := \sigma_j$ is irreducible for $j \in \{1, \dots, 8, 10, \dots, 15\}$. Then we have $\lambda_5 = \Lambda^*_{12} + \Lambda^*_{13}$ and $\lambda_5 = \underline{\sigma}_5(B_{53})\downarrow_{\tilde{A}_{15}}$, hence $\underline{\lambda}_9 := \Lambda^*_{12}$ and $\underline{\lambda}_{18} := \Lambda^*_{13}$ are irreducible Brauer characters. Moreover,

$$\underline{\sigma}_9(B_{53})\downarrow_{\tilde{A}_{15}} = -\underline{\lambda}_{12} - \underline{\lambda}_{13} + \lambda_{16} + \lambda_{17}.$$

It is clear by Table 12.5 that neither $\underline{\lambda}_{12}$ can be contained in λ_{17}, nor $\underline{\lambda}_{13}$ can be contained in $\underline{\lambda}_{16}$. Thus $\underline{\lambda}_{16} := \lambda_{16} - \underline{\lambda}_{12}$ and $\underline{\lambda}_{17} := \lambda_{17} - \underline{\lambda}_{13}$ are irreducible, and we are done. The decomposition matrix of b_{28} is Table 12.7.

Table 12.5: Mutual scalar products (b_{28})

$\langle \cdot,\cdot \rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}
λ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_4	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.
λ_5	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
λ_6	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
λ_7	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
λ_8	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.
λ_9	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
λ_{10}	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
λ_{11}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
λ_{12}	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.
λ_{13}	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_{14}	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	1	.
λ_{15}	.	.	.	.	.	1	1	.	.	.	.	.	1	.	.	.	1	.
λ_{16}	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	1
λ_{17}	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	1
λ_{18}	.	.	.	1	.	1	1	1	1	.	.	1	1	.	.	1	1	1

Table 12.6: 7-modular decomposition matrix of block B_{53} of $\tilde{\mathrm{S}}_{15}$
(defect 2, bar core (1), weight 2, content $(5,4,4,2)$)

			$[\![5,4,3,2,1]\!]$ 9152	$[\![7,4,3,1]\!]$ 193024	$[\![7,5,2,1]\!]$ 26624	$[\![7^2,1]\!]$ 128	$[\![8,4,3]\!]$ 118976	$[\![8,5,2]\!]$ 156288	$[\![8,6,1]\!]$ 704	$[\![9,5,1]\!]$ 16896	$[\![10,4,1]\!]$ 66304
⋆	128	$\langle\!\langle 15\rangle\!\rangle$	.	.	.	1	.	.	.	.	.
⋆	832	$\langle\!\langle 14,1,+\rangle\!\rangle$	.	.	.	1	.	.	1	.	.
	832	$\langle\!\langle 14,1,-\rangle\!\rangle$	.	.	.	1	.	.	1	.	.
⋆	17600	$\langle\!\langle 12,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
⋆	83200	$\langle\!\langle 11,3,1\rangle\!\rangle$	.	.	.	.	.	.	.	1	1
⋆	202176	$\langle\!\langle 10,4,1\rangle\!\rangle$	.	.	.	.	1	.	.	1	1
	292864	$\langle\!\langle 9,5,1\rangle\!\rangle$	.	.	.	.	1	1	1	1	.
⋆	27456	$\langle\!\langle 8,7,+\rangle\!\rangle$	.	.	1	1	.	.	1	.	.
	27456	$\langle\!\langle 8,7,-\rangle\!\rangle$	.	.	1	1	.	.	1	.	.
⋆	228800	$\langle\!\langle 8,6,1\rangle\!\rangle$	2	.	2	2	.	1	1	.	.
	513216	$\langle\!\langle 8,5,2\rangle\!\rangle$	2	1	1	.	1	1	.	.	.
	312000	$\langle\!\langle 8,4,3\rangle\!\rangle$	.	1	.	.	1	.	.	.	.
	228800	$\langle\!\langle 7,5,2,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
	228800	$\langle\!\langle 7,5,2,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
⋆	202176	$\langle\!\langle 7,4,3,1,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
	202176	$\langle\!\langle 7,4,3,1,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
⋆	9152	$\langle\!\langle 5,4,3,2,1\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 12.7: 7-modular decomposition matrix of block b_{28} of $\tilde{A}_{15}$
(defect 2, bar core (1), weight 2, content $(5,4,4,2)$)

			$[\![5,4,3,2,1,+]\!]$ 4576	$[\![5,4,3,2,1,-]\!]$ 4576	$[\![7,4,3,1,+]\!]$ 96512	$[\![7,4,3,1,-]\!]$ 96512	$[\![7,5,2,1,+]\!]$ 13312	$[\![7,5,2,1,-]\!]$ 13312	$[\![7^2,1,+]\!]$ 64	$[\![7^2,1,-]\!]$ 64	$[\![8,4,3,+]\!]$ 59488	$[\![8,4,3,-]\!]$ 59488	$[\![8,5,2,+]\!]$ 78144	$[\![8,5,2,-]\!]$ 78144	$[\![8,6,1,+]\!]$ 352	$[\![8,6,1,-]\!]$ 352	$[\![9,5,1,+]\!]$ 8448	$[\![9,5,1,-]\!]$ 8448	$[\![10,4,1,+]\!]$ 33152	$[\![10,4,1,-]\!]$ 33152
$\star$	64	$\langle\!\langle 15,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\star$	64	$\langle\!\langle 15,-\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\star$	832	$\langle\!\langle 14,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	1	1	.	.	.	.
$\star$	8800	$\langle\!\langle 12,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
$\star$	8800	$\langle\!\langle 12,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
$\star$	41600	$\langle\!\langle 11,3,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
$\star$	41600	$\langle\!\langle 11,3,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
$\star$	101088	$\langle\!\langle 10,4,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1
$\star$	101088	$\langle\!\langle 10,4,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	1	.
$\star$	146432	$\langle\!\langle 9,5,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	1	.	.
	146432	$\langle\!\langle 9,5,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	1	.	.	1	1	.	.	.
$\star$	27456	$\langle\!\langle 8,7\rangle\!\rangle$	.	.	.	.	1	1	1	1	.	.	.	.	1	1	.	.	.	.
$\star$	114400	$\langle\!\langle 8,6,1,+\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	1	1	.	.	.	.	.
$\star$	114400	$\langle\!\langle 8,6,1,-\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	1	.	.	1	.	.	.	.
$\star$	256608	$\langle\!\langle 8,5,2,+\rangle\!\rangle$	1	1	1	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.
	256608	$\langle\!\langle 8,5,2,-\rangle\!\rangle$	1	1	.	1	1	.	.	.	.	1	.	1	.	.	.	.	.	.
$\star$	156000	$\langle\!\langle 8,4,3,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$\star$	156000	$\langle\!\langle 8,4,3,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
	228800	$\langle\!\langle 7,5,2,1\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
	202176	$\langle\!\langle 7,4,3,1\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	4576	$\langle\!\langle 5,4,3,2,1,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	4576	$\langle\!\langle 5,4,3,2,1,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

13 The 3-modular spin characters of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

The block constants of the spin 3-blocks of $\tilde{S}_{16}$ and $\tilde{A}_{16}$ are collected in Table 13.1.

Table 13.1: Spin 3-blocks of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
9	36	14	6	(1)	5	7	36	7
10	12	3	4	$(5,2)$	3	8	12	6

The 3-modular basic spin characters of $\tilde{S}_{16}$ are $\sigma(16) := \sigma\langle\langle 16,+\rangle\rangle|_{3'}$, $\sigma(16)^a = \sigma\langle\langle 16,-\rangle\rangle|_{3'}$. The 3-modular basic spin character of $\tilde{A}_{16}$ is $\lambda(16) := \sigma(16)\downarrow_{\tilde{A}_{16}} = \lambda\langle\langle 16\rangle\rangle|_{3'}$.

13.1 The block B_{10}

The Brauer characters $\underline{\sigma}_i$ and the projective characters $\underline{\Sigma}_j$ defined by Table 13.2 satisfy $\langle \underline{\sigma}_i, \underline{\Sigma}_j\rangle = \delta_{ij}$ for all $i,j \in \{1,2,3\}$. Thus $\mathrm{IBr}(B_{10}) = \{\underline{\sigma}_1, \underline{\sigma}_2, \underline{\sigma}_3\}$. The decomposition matrix of B_{10} is Table 13.12.

Table 13.2: Some characters for B_{10}

i	$\underline{\sigma}_i$	$\deg \underline{\sigma}_i$	$\underline{\Sigma}_i$	$\deg \underline{\Sigma}_i$
1	$\sigma\langle\langle 14,2\rangle\rangle|_{3'}$	11520	$\Sigma[\![5,3^3,1,+]\!]\uparrow^{\tilde{S}_{16}}|_{B_{10}}$	21648384
2	$\sigma[\![6,4,3,2,-]\!]\uparrow^{\tilde{S}_{16}}|_{B_{10}}$	198144	$\Sigma[\![6,4,3,2,+]\!]\uparrow^{\tilde{S}_{16}}|_{B_{10}}/2$	9144576
3	$\sigma[\![7,5,2,1,+]\!]\uparrow^{\tilde{S}_{16}}|_{B_{10}}$	976896	$\Sigma[\![8,5,2]\!]\uparrow^{\tilde{S}_{16}}$	8211456

13.2 The block b_8

We have a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1,\dots,6\}$ defined by Table 13.3. The matrix of mutual scalar products is Table 13.4.

Table 13.3: Some characters for b_8

i	λ_i	$\deg\lambda_i$	Λ_i	$\deg\Lambda_i$
1	$\sigma[\![3^4,2,+]\!]\uparrow^{\tilde{A}_{16}}\mid_{b_8}$	5760	$\underline{\Sigma}_1(B_{10})\downarrow_{\tilde{A}_{16}}$	21648384
2	$\sigma[\![3^4,2,-]\!]\uparrow^{\tilde{A}_{16}}\mid_{b_8}$	5760	$\Sigma[\![3^4,2,+]\!]\uparrow^{\tilde{A}_{16}}\mid_{b_8}$	119159424
3	$\sigma[\![6,5,2,1,+]\!]\uparrow^{\tilde{A}_{16}}\mid_{b_8}$	488448	$\Lambda[\![8,5,2,+]\!]\uparrow^{\tilde{A}_{16}}$	4105728
4	$\sigma[\![6,5,2,1,-]\!]\uparrow^{\tilde{A}_{16}}\mid_{b_8}$	488448	$\Lambda[\![8,5,2,-]\!]\uparrow^{\tilde{A}_{16}}$	4105728
5	$\lambda\langle\!\langle 11,5,-\rangle\!\rangle\mid_{3'}$	104832	$\lambda(16)\otimes\Lambda[6,4,2^2,1^2,+]\mid_{b_8}$	29206656
6	$\lambda\langle\!\langle 11,5,+\rangle\!\rangle\mid_{3'}$	104832	$\lambda(16)\otimes\Lambda[6,4,2^2,1^2,-]\mid_{b_8}$	29206656

First, $\underline{\sigma}_1(B_{10})\downarrow_{\tilde{A}_{16}}=\lambda_1+\lambda_2$ and $\underline{\sigma}_3(B_{10})\downarrow_{\tilde{A}_{16}}=\lambda_3+\lambda_4$ show that $\underline{\lambda}_i:=\lambda_i$ is irreducible for $i\in\{1,\dots,4\}$. Then $\underline{\sigma}_2(B_{10})\downarrow_{\tilde{A}_{16}}=-\underline{\lambda}_1-\underline{\lambda}_2+\lambda_5+\lambda_6$ implies that either $\phi:=\lambda_5-\lambda_1$ and $\phi^c=\lambda_6-\lambda_2$, or $\psi:=\lambda_5-\lambda_2$ and $\psi^c=\lambda_6-\lambda_1$, are irreducible Brauer characters. Since $\underline{\Lambda}_3:=\Lambda_3$ and $\underline{\Lambda}_4:=\Lambda_4$ are the projective indecomposable characters corresponding to $\underline{\lambda}_3$ and $\underline{\lambda}_4$, respectively, $\Lambda_{2a}:=\Lambda_2-10\underline{\Lambda}_3-10\underline{\Lambda}_4$ improves Λ_2. Moreover, $\underline{\Lambda}_i:=\Lambda_i-3\underline{\Lambda}_3-3\underline{\Lambda}_4$ is a projective indecomposable character for $i\in\{5,6\}$. However, we do not get more information about $\mathrm{IBr}(b_8)$. In 13.3 we will use condensation and the trace formula to identify ψ and ψ^c as elements of $\mathrm{IBr}(b_8)$.

Table 13.4: Mutual scalar products (b_8)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6
λ_1	1	2	.	.	.	.
λ_2	1	1	.	.	.	.
λ_3	.	10	1	.	3	3
λ_4	.	10	.	1	3	3
λ_5	1	2	.	.	1	.
λ_6	1	2	.	.	.	1

13.3 The block b_7

We begin with basic sets $\{\lambda_1,\dots,\lambda_7\}$ and $\{\Lambda_1,\dots,\Lambda_7\}$ defined by Table 13.5. Let $\underline{\lambda}_1:=\lambda_1$. By Table 13.6, $\underline{\lambda}_2:=\lambda_2$ and $\underline{\lambda}_6:=\lambda_6$ are irreducible, and $\underline{\Lambda}_6:=\Lambda_6$ and $\underline{\Lambda}_7:=\Lambda_7$ are indecomposable.

Table 13.5: Some characters for $\mathfrak{b}_7$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(16)$	128	$\Sigma[\![3^4, 2, +]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	104789376
2	$\lambda[\![4, 3^3, 2]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	768	$\Lambda[\![4, 3^3, 2]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	64198656
3	$\lambda\langle\!\langle 13, 3, -\rangle\!\rangle\|_{3'}$	22400	$\Lambda[\![5, 3^2, 2, 1 : 2]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}/6$	13436928
4	$\lambda\langle\!\langle 12, 4, -\rangle\!\rangle\|_{3'}$	58240	$\Sigma[\![5, 4, 3, 2, -]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	53934336
5	$\lambda\langle\!\langle 9, 7, -\rangle\!\rangle\|_{3'}$	91520	$\Lambda[\![6, 5, 3, 1]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}/3$	15116544
6	$\lambda[\![6, 4, 3, 2]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	228864	$\Lambda[\![7, 4, 3, 1, +]\!]\uparrow^{\tilde{A}_{16}}$	4852224
7	$\lambda\langle\!\langle 8, 7, 1\rangle\!\rangle\|_{3'}$	256256	$\lambda(16) \otimes \Lambda[7, 5, 3, 1]\ \|_{\mathfrak{b}_7}/2$	10264320
8	$\lambda(16) \otimes \lambda[13, 3]$	43136	$\underline{\Lambda}_6(\mathfrak{b}_8) \otimes \lambda[15, 1]\ \|_{\mathfrak{b}_7}$	30233088
9	$\lambda(16) \otimes \lambda[10, 6]$	182784	$\Lambda[\![3^4, 2, 1]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	123918336
10	$\lambda(16) \otimes \lambda[11, 5]\ \|_{\mathfrak{b}_7}$	71680		
11	$\lambda[\![7, 5, 2, 1]\!]\uparrow^{\tilde{A}_{16}}\|_{\mathfrak{b}_7}$	327680		

Table 13.6: Mutual scalar products ($\mathfrak{b}_7$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7
λ_1	1	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.
λ_3	3	1	1	1	.	.	.
λ_4	3	1	1	2	.	.	.
λ_5	3	.	.	1	1	.	.
λ_6	.	.	.	.	.	1	.
λ_7	4	2	.	1	1	.	1

As $\underline{\lambda}_6$ corresponds to $\underline{\Lambda}_6$ and $\langle\underline{\lambda}_6, \Lambda_8\rangle = 1$, we improve the basic set of projective characters by swapping Λ_4 and $\Lambda_{8a} := \Lambda_8 - \underline{\Lambda}_6$. Then $\underline{\Lambda}_4 := \Lambda_{8a} = \lambda_4^*$ is indecomposable. The relations $\lambda_8 = -\underline{\lambda}_1 - 2\underline{\lambda}_2 + 2\lambda_3$ and $\lambda_9 = -2\underline{\lambda}_1 + 2\lambda_5$ yield $\lambda_{3a} := \lambda_3 - \underline{\lambda}_1 - \underline{\lambda}_2$ and $\lambda_{5a} := \lambda_5 - \underline{\lambda}_1$. It follows from $\lambda_{10} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - 2\lambda_{3a} + 2\lambda_4$ and the decomposition of λ_{3a} in terms of the Brauer atoms (see Table 13.7) that λ_4 may be improved by $\lambda_{4a} := \lambda_4 + \underline{\lambda}_1 - \underline{\lambda}_2 - \lambda_{3a}$. Decomposing λ_{10} again yields $\lambda_{10} = -4\underline{\lambda}_1 + 2\lambda_{4a}$ and thus $\lambda_{4b} := \lambda_{4a} - 2\underline{\lambda}_1$. Similarly, $\lambda_{11} = -4\underline{\lambda}_1 - 2\underline{\lambda}_2 - 2\lambda_{5a} + 2\lambda_7$ gives $\lambda_{7a} := \lambda_7 - \underline{\lambda}_2 - \lambda_{5a}$ at

Table 13.7: Mutual scalar products (b_7)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	$\underline{\Lambda}_4$	Λ_5	$\underline{\Lambda}_6$	$\underline{\Lambda}_7$
$\underline{\lambda}_1$	1	.	.	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.	.	.
λ_{3a}	2	.	1	.	.	.	.
λ_4	3	1	1	1	.	.	.
λ_{5a}	2	.	.	.	1	.	.
$\underline{\lambda}_6$	.	.	.	.	.	1	.
λ_7	4	2	.	.	1	.	1

first, and $\lambda_{11} = -4\underline{\lambda}_1 + 2\lambda_{7a}$ yields $\lambda_{7b} := \lambda_{7a} - 2\underline{\lambda}_1$. Now $\underline{\Lambda}_3 := \Lambda_3$ and $\underline{\Lambda}_5 := \Lambda_5$ are projective atoms, hence indecomposable. Since $\Lambda_9 = 2\Lambda_1 - 3\underline{\Lambda}_3 - 3\underline{\Lambda}_5$, we improve Λ_1 by $\Lambda_{1a} := \Lambda_1 - 2\underline{\Lambda}_3 - 2\underline{\Lambda}_5$. It follows from Table 13.8 that the latter is indecomposable, and $\underline{\lambda}_3 := \lambda_{3a}$, $\underline{\lambda}_4 := \lambda_{4b}$, and $\underline{\lambda}_5 := \lambda_{5a}$ are irreducible.

Table 13.8: Mutual scalar products (b_7)

$\langle\cdot,\cdot\rangle$	Λ_{1a}	Λ_2	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	$\underline{\Lambda}_5$	$\underline{\Lambda}_6$	$\underline{\Lambda}_7$
$\underline{\lambda}_2$	.	1	.	.	.	.	.
λ_{7b}	.	1	.	.	.	.	1

We use condensation to complete $\mathrm{IBr}(b_8)$, see 13.2. After that, we will wait until we have a close approximation of $\mathrm{IBr}(B_9)$ (see 13.4) before we will answer the question whether or not λ_{7b} is irreducible. As before, we take $K := \langle t_{(7^2,1^2)}, t_{(7,1^9)}\rangle \cong 7^2$ with corresponding trivial idempotent e. Let $x := (1,7,13,3,9,15,16,4,10)(2,8,14,6,12,5,11) \in A_{16}$. Then

$$|K|^{-1}\sum_{k\in K}\overline{\breve{\varphi}(\tilde{x}k)} \equiv \begin{cases} \zeta_9^5 & \text{if } \varphi = \phi, \\ \zeta_9^7 & \text{if } \varphi = \phi^c, \\ 1_F & \text{if } \varphi = \psi, \\ 1_F & \text{if } \varphi = \psi^c. \end{cases}$$

Let $15a^+$ denote the reduced permutation module of S_{16} over $GF(3)$. We chop the sequence $(15a^+ \otimes 15a^+, 15a^+ \otimes 103a^+, 15a^+ \otimes 337a^+, 15a^+ \otimes 1260a^+)$ to obtain a module

$2108a^+$ of S_{16} over GF(3) that affords $\sigma[11,5]$. Let $128a^-$ denote a basic spin module of $\tilde{S}_{16}$ over $F := \mathrm{GF}(9)$ with Brauer character $\sigma(16)$. We choose $a := t_1t_2$ and $b := t_{15}t_{14}\cdots t_2$ as generators for $\tilde{A}_{16}$. Noting that both, $\sigma(16)$ and $\sigma[11,5]$, restrict irreducibly to A_{16}, we condense the $F\tilde{S}_{16}$-module $V := 128a^- \otimes 2108a^+$ with respect to $c := (ab)^3b$ and $ab(ab^2)^2c$. The decomposition of the corresponding Brauer character

$$\lambda(16) \otimes \lambda[11,5] = -\underline{\lambda}_1(b_8) - \underline{\lambda}_2(b_8) + 2\underline{\lambda}_4(b_7) + \lambda_5(b_8) + \lambda_6(b_8)$$

shows that V has two distinct constituents, say X_5 and X_6, that afford the irreducible, conjugate constituents of $\lambda_5(b_8)$ and $\lambda_6(b_8)$. We have $\dim Ve = 5320$ and $\dim Xe = 1956$ for $X \in \{X_5, X_6\}$. Then $\mathrm{tr}_{Xe}(e\tilde{x}e) \equiv 1_F$, thus X_5 and X_6 correspond to ψ and ψ^c (see 13.2). The decomposition matrix of b_8 is Table 13.13.

13.4 The block B_9

We have basic sets $\{\sigma_1, \ldots, \sigma_{14}\}$ and $\{\Sigma_1, \ldots, \Sigma_{14}\}$ defined by Table 13.9.

Table 13.9: Some characters for B_9

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(16)$	128	$\underline{\Lambda}_7(b_7)\uparrow^{\tilde{S}_{16}}$	95364864
2	$\sigma(16)^a$	128	$\sigma(16) \otimes \Sigma[11,3,2] \mid_{B_9}$	422050176
3	$\sigma[\![4,3^3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	768	$\Sigma[\![4,3^3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	64198656
4	$\sigma[\![4,3^3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	768	$\Sigma[\![4,3^3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	64198656
5	$\sigma\langle\!\langle 13,2,1,+\rangle\!\rangle\mid_{3'}$	22528	$\Sigma[\![5,3^3,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	40310784
6	$\sigma\langle\!\langle 13,2,1,-\rangle\!\rangle\mid_{3'}$	22528	$\Sigma[\![5,3^2,2,1:2]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}/3$	26873856
7	$\sigma(16) \otimes \sigma[11,5] \mid_{B_9}$	71680	$\Sigma[\![6,4,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	30233088
8	$\sigma[\![3^3,2,1:3,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	104960	$\Sigma[\![6,4,3,1:2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	54494208
9	$\sigma[\![6,4,3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	228864	$\Sigma[\![6,4,3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	30233088
10	$\sigma[\![6,4,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	228864	$\Sigma[\![6,4,1,+:3,2]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	192222720
11	$\sigma\langle\!\langle 8,7,1,-\rangle\!\rangle\mid_{3'}$	256256	$\Sigma[\![7,4,3,1]\!]\uparrow^{\tilde{S}_{16}}$	9704448
12	$\sigma\langle\!\langle 8,7,1,+\rangle\!\rangle\mid_{3'}$	256256	$\Sigma[\![7,5,2,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	20528640
13	$\sigma[\![3^3,1,+:3,2,1]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	291328	$\Sigma[\![6,5,2,1:2]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}/3$	30233088
14	$\sigma(16) \otimes \sigma[10,6]$	182784	$\Sigma[\![6,5,3,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_9}$	45349632
15	$\sigma[\![6,5,3,1,+]\!]\uparrow^{\tilde{S}_{16}}$	438272		
16	$\sigma[\![6,5,3,1,-]\!]\uparrow^{\tilde{S}_{16}}$	438272		

Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$. We have $\underline{\lambda}_2 \uparrow^{\tilde{S}_{16}} = \sigma_3 + \sigma_4$ and $\underline{\lambda}_6(b_7) \uparrow^{\tilde{S}_{16}} = \sigma_9 + \sigma_{10}$, hence $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{3, 4, 9, 10\}$. Then

$$\underline{\lambda}_3(b_7) \uparrow^{\tilde{S}_{16}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_5 + \sigma_6$$

and the decompositions of $\underline{\sigma}_3$, $\underline{\sigma}_4$, σ_5, and σ_6 in terms of the Brauer atoms (see Table 13.10) show that $\underline{\sigma}_3$ is contained in σ_5, and $\underline{\sigma}_4$ is contained in σ_6. It also follows that there are suitable $x_1, x_2 \in \{0, 1, 2\}$ such that

$$\sigma_5 - \underline{\sigma}_3 - x_1\underline{\sigma}_1 - x_2\underline{\sigma}_2 \quad \text{and} \quad \sigma_6 - \underline{\sigma}_4 - (2 - x_1)\underline{\sigma}_1 - (2 - x_2)\underline{\sigma}_2$$

are irreducible. As $(\sigma_5 - \underline{\sigma}_3 - x_1\underline{\sigma}_1 - x_2\underline{\sigma}_2)^{\circledast} = \sigma_6 - \underline{\sigma}_4 - x_1\underline{\sigma}_1 - x_2\underline{\sigma}_2$, we conclude $x_1 = x_2 = 1$ and denote the resulting irreducible Brauer characters by $\underline{\sigma}_5$ and $\underline{\sigma}_6$. Similarly, we deduce from

$$\lambda_{7b}(b_7) \uparrow^{\tilde{S}_{16}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_{11} + \sigma_{12} - \sigma_{14}$$

and $\sigma_{11}^{\circledast} = \sigma_{12}$ that $\underline{\sigma}_1 + \underline{\sigma}_2$ is contained in both, σ_{11} and σ_{12}. Note that Table 13.10 does not show whether $\underline{\sigma}_3$ is contained in σ_{11} and $\underline{\sigma}_4$ is contained in σ_{12}, or the other way around. Nevertheless, we can improve σ_i by $\sigma_{ia} := \sigma_i - \underline{\sigma}_1 - \underline{\sigma}_2$ for $i \in \{11, 12\}$.

Table 13.10: Mutual scalar products (B_9)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	1	1	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.	.	.	.	.	.
σ_4	.	.	.	1	.	.	.	.	.	.	.	.	.	.
σ_5	2	3	1	.	1	1	.	.	.	.	.	.	.	.
σ_6	2	3	.	1	2	1	.	.	.	.	.	.	.	.
σ_7	.	4	.	.	.	.	1	1	1	1	.	.	.	.
σ_8	6	11	3	2	4	3	1	.	.	1	.	.	.	.
σ_9	.	10	.	.	.	.	.	3	1	3	1	.	.	.
σ_{10}	.	10	.	.	.	.	1	3	.	3	1	.	.	.
σ_{11}	2	6	1	1	.	.	.	.	.	6	.	1	1	2
σ_{12}	2	6	1	1	.	.	.	.	.	6	.	1	1	1
σ_{13}	8	20	3	4	6	4	1	2	2	3	.	.	1	2
σ_{14}	.	4	.	.	.	.	.	.	.	3	.	.	2	3

We turn to the basic set of projective characters. Since $\underline{\Lambda}_6(b_7)\uparrow^{\tilde{S}_{16}} = \Sigma_{11} = \sigma_9^* + \sigma_{10}^*$ and $\underline{\Lambda}_7(b_7)\uparrow^{\tilde{S}_{16}} = \Sigma_{12} = \sigma_{11}^* + \sigma_{12}^*$, the projective atoms $\underline{\Sigma}_i := \sigma_i^*$, $i \in \{9, \dots, 12\}$, are projective indecomposable characters. Now $\underline{\sigma}_i$ corresponds to $\underline{\Sigma}_i$ for $i \in \{9, 10\}$, hence $\Sigma_{2a} := \Sigma_2 - 10\underline{\Sigma}_9 - 10\underline{\Sigma}_{10}$ improves Σ_2. In addition, $\underline{\Lambda}_4(b_7)\uparrow^{\tilde{S}_{16}} = \Sigma_7 + \Sigma_9 - \Sigma_{11}$ and the decompositions of Σ_7 and Σ_9 in terms of the projective atoms imply that $\underline{\Sigma}_7 := \Sigma_7 - \underline{\Sigma}_{10}$ and $\underline{\Sigma}_8 := \Sigma_9 - \underline{\Sigma}_9$ are indecomposable. The mutual scalar products of the improved basic sets are given in Table 13.11.

Table 13.11: Mutual scalar products (B_9)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_{2a}	Σ_3	Σ_4	Σ_5	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	$\underline{\Sigma}_{11}$	$\underline{\Sigma}_{12}$	Σ_{13}	Σ_{14}
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_3$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_5$	.	2	.	.	1	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_6$	.	2	.	.	2	1	.	.	.	.	.	.	.	.
σ_7	.	4	.	.	.	.	1	1	.	.	.	.	.	.
σ_8	6	11	3	2	4	3	1	.	.	.	.	.	.	.
$\underline{\sigma}_9$	.	.	.	.	.	.	.	.	1	.	.	.	.	.
$\underline{\sigma}_{10}$	.	.	.	.	.	.	.	.	.	1	.	.	.	.
σ_{11a}	.	5	1	1	.	.	.	.	.	.	1	.	1	2
σ_{12a}	.	5	1	1	.	.	.	.	.	.	.	1	1	1
σ_{13}	8	20	3	4	6	4	1	2	.	.	.	.	1	2
σ_{14}	.	4	.	.	.	.	.	.	.	.	.	.	2	3

We have $\underline{\lambda}_4(b_7)\uparrow^{\tilde{S}_{16}} = \sigma_7 = 4\Sigma_{2a}^* + \underline{\Sigma}_7^* + \underline{\Sigma}_8^*$ with $\deg \Sigma_{2a}^* = 0$ and $\deg \underline{\Sigma}_7^* = \deg \underline{\Sigma}_8^* = 35840$. Thus the irreducible constituents of σ_7 are $x\Sigma_{2a}^* + \underline{\Sigma}_7^*$ and $(4-x)\underline{\Sigma}_7^* + \underline{\Sigma}_8^*$ for some suitable $x \in \{0, \dots, 4\}$. By (1.2), there are only two elements of $\mathrm{IBr}(B_9)$ that are fixed under complex conjugation. We know these already, namely $\underline{\sigma}_1$ and $\underline{\sigma}_2$. This forces $(x\Sigma_{2a}^* + \underline{\Sigma}_7^*)^{\circledast} = (x\Sigma_{2a}^* + \underline{\Sigma}_7^*)^{a}$ which is fulfilled only for $x = 2$. Thus $\underline{\sigma}_7 := 2\Sigma_{2a}^* + \underline{\Sigma}_7^*$ and $\underline{\sigma}_8 := 2\Sigma_{2a}^* + \underline{\Sigma}_8^*$ are irreducible Brauer characters. Similarly, we conclude from $\underline{\lambda}_5(b_7)\uparrow^{\tilde{S}_{16}} = \sigma_{14} = 4\Sigma_{2a}^* + 2\Sigma_{13}^* + 3\Sigma_{14}^*$ and $\lambda_{7b}(b_7)\uparrow^{\tilde{S}_{16}} = -\underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_{11a} + \sigma_{12a} - \sigma_{14}$ that the irreducible constituents of σ_{14} are $x\Sigma_{2a}^* + \Sigma_{13}^* + \Sigma_{14}^*$ and $(4-x)\Sigma_{2a}^* + \Sigma_{13}^* + 2\Sigma_{14}^*$ for a suitable $x \in \{0, \dots, 4\}$. It follows that $x = 2$. Hence $\underline{\sigma}_{13} := 2\Sigma_{2a}^* + \Sigma_{13}^* + \Sigma_{14}^*$ and $\underline{\sigma}_{14} := 2\Sigma_{2a}^* + \Sigma_{13}^* + 2\Sigma_{14}^*$ are irreducible Brauer characters. Furthermore, $\sigma_{11b} :=$

$\sigma_{11a} - \underline{\sigma}_{14}$ and $\sigma_{12b} := \sigma_{12a} - \underline{\sigma}_{13}$ improve σ_{11a} and σ_{12a}, respectively. We still have to subtract one element of $\{\underline{\sigma}_3, \underline{\sigma}_4\}$ from σ_{11b} (and the other one from σ_{12b}) because $\lambda_{7b}(b_7)\uparrow^{\tilde{S}_{16}} = -\underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_{11b} + \sigma_{12b}$. This is controlled by

$$\sigma_{15} = \underline{\sigma}_1 + \underline{\sigma}_2 - \underline{\sigma}_3 + \sigma_{11b} + \underline{\sigma}_{13} + 2\underline{\sigma}_{14},$$
$$\sigma_{16} = \underline{\sigma}_1 + \underline{\sigma}_2 - \underline{\sigma}_4 + \sigma_{12b} + 2\underline{\sigma}_{13} + \underline{\sigma}_{14}.$$

We get $\sigma_{11c} := \sigma_{11b} - \underline{\sigma}_3$ and $\sigma_{12c} := \sigma_{12b} - \underline{\sigma}_4$. with $\sigma_{11c}\downarrow_{\tilde{A}_{16}} = \sigma_{12c}\downarrow_{\tilde{A}_{16}} = \lambda_{7b}(b_7)$. Let $768a^-$ denote an irreducible 768-dimensional constituent of $128a^- \otimes 15a^+$. Then $W := 768a^- \otimes 2108a^+$ has a composition factor X with Brauer character φ such that either $\varphi \in \{\sigma_{11c}, \sigma_{12c}\}$ or $\varphi \in \{\sigma_{11c} - \underline{\sigma}_4, \sigma_{12c} - \underline{\sigma}_3\}$. Let $\hat{K} := \langle (\hat{t}_{(7,7,1^2)})^2, (\hat{t}_{(7,1^9)})^2, \hat{t}_{15}\rangle \leqslant \hat{S}_{16}$ with $\hat{t}_{(7,7,1^2)}$ and $\hat{t}_{(7,1^9)}$ defined for the double cover $\hat{S}_{16} = \langle \hat{t}_1, \ldots, \hat{t}_{15}\rangle \ncong \tilde{S}_{16}$ of S_{16} analogously as in (5.1.2). In particular, $\hat{t}_{15}$ is of order 2 and $\hat{K} \cong 7^2 \times 2$. Let $a := \hat{t}_1$, $b := \hat{t}_{15}\hat{t}_{14}\cdots\hat{t}_1$, $c := (ab)^3 b$ and $d := ab(ab^2)^2 c$. The $F\hat{S}_{16}$-module $\hat{V}$ corresponding to V is realizable over $F := GF(3)$. Our condensation algebra is $C := \langle ece, ede\rangle_F$ where $e := e_{\hat{K}}$. We know $\dim \hat{X}_3 = \dim \hat{X}_4 = 12$ for the $F\hat{S}_{16}$-modules $\hat{X}_3$ and $\hat{X}_4$ that induce $F\tilde{S}_{16}$-modules X_3 and X_4 affording $\underline{\sigma}_3$ and $\underline{\sigma}_4$, respectively. We expect $\dim \hat{X}e = 1676$ or $\dim \hat{X}e = 1676 - 12$ where $\hat{X}$ denotes the $F\hat{S}_{16}$-module that induces X. We obtain $\dim \hat{X}e\downarrow_C = 1676$. Thus $\underline{\sigma}_{11} := \sigma_{11c}$, $\underline{\sigma}_{12} := \sigma_{12c}$, and $\underline{\lambda}_7(b_7) := \lambda_{7b}$ are irreducible. The decomposition matrices of b_7 and B_9 are Tables 13.14 and 13.15, respectively.

Table 13.12: 3-modular decomposition matrix of block B_{10} of $\tilde{S}_{16}$ (defect 4, bar core $(5,2)$, weight 3, content $(10,6)$)

			$[\![5,3^3,2]\!]$ 11520	$[\![6,5,3,2]\!]$ 198144	$[\![8,5,2,1]\!]$ 976896
$\star$	11520	$\langle\!\langle 14,2\rangle\!\rangle$	1	.	.
$\star$	209664	$\langle\!\langle 11,5\rangle\!\rangle$	1	1	.
	221184	$\langle\!\langle 11,3,2,+\rangle\!\rangle$	2	1	.
	221184	$\langle\!\langle 11,3,2,-\rangle\!\rangle$	2	1	.
	1198080	$\langle\!\langle 9,5,2,+\rangle\!\rangle$	2	1	1
	1198080	$\langle\!\langle 9,5,2,-\rangle\!\rangle$	2	1	1
$\star$	988416	$\langle\!\langle 8,6,2,+\rangle\!\rangle$	1	.	1
	988416	$\langle\!\langle 8,6,2,-\rangle\!\rangle$	1	.	1
	1209600	$\langle\!\langle 8,5,3,+\rangle\!\rangle$	3	1	1
	1209600	$\langle\!\langle 8,5,3,-\rangle\!\rangle$	3	1	1
	1419264	$\langle\!\langle 8,5,2,1\rangle\!\rangle$	4	2	1
	419328	$\langle\!\langle 6,5,3,2\rangle\!\rangle$	2	2	.

Table 13.13: 3-modular decomposition matrix of block $\mathfrak{b}_8$ of $\tilde{A}_{16}$ (defect 4, bar core $(5,2)$, weight 3, content $(10,6)$)

			$[\![5,3^3,2,+]\!]$ 5760	$[\![5,3^3,2,-]\!]$ 5760	$[\![6,5,3,2,+]\!]$ 99072	$[\![6,5,3,2,-]\!]$ 99072	$[\![8,5,2,1,+]\!]$ 488448	$[\![8,5,2,1,-]\!]$ 488448
⋆	5760	$\langle\!\langle 14,2,+\rangle\!\rangle$	.	1	.	.	.	.
⋆	5760	$\langle\!\langle 14,2,-\rangle\!\rangle$	1	.	.	.	.	.
⋆	104832	$\langle\!\langle 11,5,+\rangle\!\rangle$	1	.	.	1	.	.
⋆	104832	$\langle\!\langle 11,5,-\rangle\!\rangle$	.	1	1	.	.	.
	221184	$\langle\!\langle 11,3,2\rangle\!\rangle$	2	2	1	1	.	.
	1198080	$\langle\!\langle 9,5,2\rangle\!\rangle$	2	2	1	1	1	1
	988416	$\langle\!\langle 8,6,2\rangle\!\rangle$	1	1	.	.	1	1
	1209600	$\langle\!\langle 8,5,3\rangle\!\rangle$	3	3	1	1	1	1
⋆	709632	$\langle\!\langle 8,5,2,1,+\rangle\!\rangle$	2	2	1	1	.	1
⋆	709632	$\langle\!\langle 8,5,2,1,-\rangle\!\rangle$	2	2	1	1	1	.
	209664	$\langle\!\langle 6,5,3,2,+\rangle\!\rangle$	1	1	1	1	.	.
	209664	$\langle\!\langle 6,5,3,2,-\rangle\!\rangle$	1	1	1	1	.	.

Table 13.14: 3-modular decomposition matrix of block $\mathfrak{b}_7$ of $\tilde{A}_{16}$ (defect 6, bar core (1), weight 5, content (11, 5))

			$[\![3^5,1]\!]$ 128	$[\![4,3^3,2,1]\!]$ 768	$[\![5,4,3^2,1]\!]$ 21504	$[\![6,4,3,2,1]\!]$ 35840	$[\![6,5,4,1]\!]$ 91392	$[\![7,4,3,2]\!]$ 228864	$[\![7,5,3,1]\!]$ 163840
⋆	128	$\langle\!\langle 16\rangle\!\rangle$	1	.	.	.	.	.	.
⋆	896	$\langle\!\langle 15,1,+\rangle\!\rangle$	1	1	.	.	.	.	.
	896	$\langle\!\langle 15,1,-\rangle\!\rangle$	1	1	.	.	.	.	.
⋆	22400	$\langle\!\langle 13,3,+\rangle\!\rangle$	1	1	1	.	.	.	.
	22400	$\langle\!\langle 13,3,-\rangle\!\rangle$	1	1	1	.	.	.	.
	22528	$\langle\!\langle 13,2,1\rangle\!\rangle$	2	1	1	.	.	.	.
⋆	58240	$\langle\!\langle 12,4,+\rangle\!\rangle$	1	1	1	1	.	.	.
	58240	$\langle\!\langle 12,4,-\rangle\!\rangle$	1	1	1	1	.	.	.
	118272	$\langle\!\langle 12,3,1\rangle\!\rangle$	4	4	2	2	.	.	.
	326144	$\langle\!\langle 11,4,1\rangle\!\rangle$	2	5	1	2	.	1	.
	128128	$\langle\!\langle 10,6,+\rangle\!\rangle$	1	1	.	1	1	.	.
	128128	$\langle\!\langle 10,6,-\rangle\!\rangle$	1	1	.	1	1	.	.
	559104	$\langle\!\langle 10,5,1\rangle\!\rangle$	2	4	.	2	1	1	1
	732160	$\langle\!\langle 10,4,2\rangle\!\rangle$	6	7	2	3	2	1	1
	150528	$\langle\!\langle 10,3,2,1,+\rangle\!\rangle$	2	2	1	1	1	.	.
	150528	$\langle\!\langle 10,3,2,1,-\rangle\!\rangle$	2	2	1	1	1	.	.
⋆	91520	$\langle\!\langle 9,7,+\rangle\!\rangle$	1	.	.	.	1	.	.
	91520	$\langle\!\langle 9,7,-\rangle\!\rangle$	1	.	.	.	1	.	.
	585728	$\langle\!\langle 9,6,1\rangle\!\rangle$	4	4	.	2	2	.	2
	704000	$\langle\!\langle 9,4,3\rangle\!\rangle$	8	8	2	4	2	.	2
	501760	$\langle\!\langle 9,4,2,1,+\rangle\!\rangle$	6	5	2	3	2	.	1
	501760	$\langle\!\langle 9,4,2,1,-\rangle\!\rangle$	6	5	2	3	2	.	1
⋆	256256	$\langle\!\langle 8,7,1\rangle\!\rangle$	2	1	.	.	1	.	1
	582400	$\langle\!\langle 8,4,3,1,+\rangle\!\rangle$	4	6	1	2	1	1	1
	582400	$\langle\!\langle 8,4,3,1,-\rangle\!\rangle$	4	6	1	2	1	1	1
	630784	$\langle\!\langle 7,6,3\rangle\!\rangle$	8	6	2	2	2	.	2
	352000	$\langle\!\langle 7,6,2,1,+\rangle\!\rangle$	4	4	1	2	1	.	1
	352000	$\langle\!\langle 7,6,2,1,-\rangle\!\rangle$	4	4	1	2	1	.	1
	465920	$\langle\!\langle 7,5,4\rangle\!\rangle$	6	5	2	2	2	.	1
	768768	$\langle\!\langle 7,5,3,1,+\rangle\!\rangle$	6	8	2	4	2	1	1
	768768	$\langle\!\langle 7,5,3,1,-\rangle\!\rangle$	6	8	2	4	2	1	1
⋆	266240	$\langle\!\langle 7,4,3,2,+\rangle\!\rangle$	.	2	.	1	.	1	.
	266240	$\langle\!\langle 7,4,3,2,-\rangle\!\rangle$	.	2	.	1	.	1	.
	186368	$\langle\!\langle 6,5,4,1,+\rangle\!\rangle$	2	2	1	2	1	.	.
	186368	$\langle\!\langle 6,5,4,1,-\rangle\!\rangle$	2	2	1	2	1	.	.
	73216	$\langle\!\langle 6,4,3,2,1\rangle\!\rangle$	.	2	.	2	.	.	.

Table 13.15: 3-modular decomposition matrix of block B_9 of $\tilde{S}_{16}$ (defect 6, bar core (1), weight 5, content $(11, 5)$)

			$[\![3^5,1,+]\!]$ 128	$[\![3^5,1,-]\!]$ 128	$[\![4,3^3,2,1,+]\!]$ 768	$[\![4,3^3,2,1,-]\!]$ 768	$[\![5,4,3^2,1,+]\!]$ 21504	$[\![5,4,3^2,1,-]\!]$ 21504	$[\![6,4,3,2,1,+]\!]$ 35840	$[\![6,4,3,2,1,-]\!]$ 35840	$[\![6,5,4,1,+]\!]$ 91392	$[\![6,5,4,1,-]\!]$ 91392	$[\![7,4,3,2,+]\!]$ 228864	$[\![7,4,3,2,-]\!]$ 228864	$[\![7,5,3,1,+]\!]$ 163840	$[\![7,5,3,1,-]\!]$ 163840
⋆	128	$\langle\!\langle 16,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	128	$\langle\!\langle 16,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1792	$\langle\!\langle 15,1\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.
	44800	$\langle\!\langle 13,3\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.
⋆	22528	$\langle\!\langle 13,2,1,+\rangle\!\rangle$	1	1	1	.	.	1	.	.	.	.	.	.	.	.
⋆	22528	$\langle\!\langle 13,2,1,-\rangle\!\rangle$	1	1	.	1	1	.	.	.	.	.	.	.	.	.
	116480	$\langle\!\langle 12,4\rangle\!\rangle$	1	1	1	1	1	1	1	1	.	.	.	.	.	.
	118272	$\langle\!\langle 12,3,1,+\rangle\!\rangle$	2	2	2	2	1	1	1	1	.	.	.	.	.	.
	118272	$\langle\!\langle 12,3,1,-\rangle\!\rangle$	2	2	2	2	1	1	1	1	.	.	.	.	.	.
⋆	326144	$\langle\!\langle 11,4,1,+\rangle\!\rangle$	1	1	3	2	.	1	1	1	.	.	.	1	.	.
⋆	326144	$\langle\!\langle 11,4,1,-\rangle\!\rangle$	1	1	2	3	1	.	1	1	.	.	1	.	.	.
	256256	$\langle\!\langle 10,6\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	1	.	.	.	.
⋆	559104	$\langle\!\langle 10,5,1,+\rangle\!\rangle$	1	1	2	2	.	.	1	1	.	1	.	1	1	.
	559104	$\langle\!\langle 10,5,1,-\rangle\!\rangle$	1	1	2	2	.	.	1	1	1	.	1	.	.	1
⋆	732160	$\langle\!\langle 10,4,2,+\rangle\!\rangle$	3	3	4	3	1	1	2	1	1	1	1	.	.	1
	732160	$\langle\!\langle 10,4,2,-\rangle\!\rangle$	3	3	3	4	1	1	1	2	1	1	.	1	1	.
	301056	$\langle\!\langle 10,3,2,1\rangle\!\rangle$	2	2	2	2	1	1	1	1	1	1	.	.	.	.
⋆	183040	$\langle\!\langle 9,7\rangle\!\rangle$	1	1	.	.	.	.	.	.	1	1	.	.	.	.
	585728	$\langle\!\langle 9,6,1,+\rangle\!\rangle$	2	2	2	2	.	.	1	1	1	1	.	.	1	1
	585728	$\langle\!\langle 9,6,1,-\rangle\!\rangle$	2	2	2	2	.	.	1	1	1	1	.	.	1	1
	704000	$\langle\!\langle 9,4,3,+\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
	704000	$\langle\!\langle 9,4,3,-\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
	1003520	$\langle\!\langle 9,4,2,1\rangle\!\rangle$	6	6	5	5	2	2	3	3	2	2	.	.	1	1
⋆	256256	$\langle\!\langle 8,7,1,+\rangle\!\rangle$	1	1	.	1	.	.	.	.	.	1	.	.	1	.
⋆	256256	$\langle\!\langle 8,7,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	1	.	.	.	.	1
	1164800	$\langle\!\langle 8,4,3,1\rangle\!\rangle$	4	4	6	6	1	1	2	2	1	1	1	1	1	1
	630784	$\langle\!\langle 7,6,3,+\rangle\!\rangle$	4	4	3	3	1	1	1	1	1	1	.	.	1	1
	630784	$\langle\!\langle 7,6,3,-\rangle\!\rangle$	4	4	3	3	1	1	1	1	1	1	.	.	1	1
	704000	$\langle\!\langle 7,6,2,1\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
⋆	465920	$\langle\!\langle 7,5,4,+\rangle\!\rangle$	3	3	3	2	1	1	1	1	1	1	.	.	.	1
	465920	$\langle\!\langle 7,5,4,-\rangle\!\rangle$	3	3	2	3	1	1	1	1	1	1	.	.	1	.
	1537536	$\langle\!\langle 7,5,3,1\rangle\!\rangle$	6	6	8	8	2	2	4	4	2	2	1	1	1	1
	532480	$\langle\!\langle 7,4,3,2\rangle\!\rangle$	.	.	2	2	.	.	1	1	.	.	1	1	.	.
	372736	$\langle\!\langle 6,5,4,1\rangle\!\rangle$	2	2	2	2	1	1	2	2	1	1	.	.	.	.
⋆	73216	$\langle\!\langle 6,4,3,2,1,+\rangle\!\rangle$	.	.	1	1	.	.	1	1	.	.	.	.	.	.
	73216	$\langle\!\langle 6,4,3,2,1,-\rangle\!\rangle$	.	.	1	1	.	.	1	1	.	.	.	.	.	.

14 The 5-modular spin characters of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

Both $\tilde{S}_{16}$ and $\tilde{A}_{16}$ have three spin 5-blocks of defect 0. The block constants of all other spin 5-blocks of are given in Table 14.1.

Table 14.1: Spin 5-blocks of non-zero defect of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
30	30	20	3	(1)	3	19	27	10
31	11	5	2	$(4,2)$	2	20	13	10
33	4	2	1	$(8,3)$	1	23	5	4

The 5-modular basic spin characters of $\tilde{S}_{16}$ are $\sigma(16) := \sigma\langle\langle 16,+\rangle\rangle|_{5'}$, $\sigma(16)^a = \sigma\langle\langle 16,-\rangle\rangle|_{5'}$. They restrict irreducibly to the 5-modular basic spin character $\lambda(16) := \lambda\langle\langle 16\rangle\rangle|_{5'}$ of $\tilde{A}_{16}$.

14.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{33} and b_{23} are given in Appendix A as Figures A.17 and A.18, respectively.

14.2 The block b_{19}

Consider the Brauer characters λ_i and the projective characters Λ_i defined by Table 14.2. The FBA algorithm (2.11) creates from input $(\lambda_1, \dots, \lambda_{11})$ a Brauer character $\lambda_{5a} := (\lambda_1 + 2\lambda_2 + \lambda_5)/2$ such that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_{5a}, \lambda_6, \lambda_7, \lambda_9, \lambda_{10}, \lambda_{11}\}$ is a basic set of Brauer characters. The mutual scalar products with respect to the basic set of projective characters, $\{\Lambda_1, \dots, \Lambda_{10}\}$, are given in Table 14.3. Clearly, $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1,2,3,4,6,10\}$, and $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{6,\dots,10\}$.
Then $\lambda_{12} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \underline{\lambda}_4 + \lambda_{5a} + \underline{\lambda}_{10}$, $\lambda_{13} = -\underline{\lambda}_1 - 2\underline{\lambda}_2 + 2\lambda_7$, and $\lambda_{14} = -\underline{\lambda}_4 + \underline{\lambda}_6 + \lambda_9$ lead to $\underline{\lambda}_5 := \lambda_{5a} - \underline{\lambda}_1 - \underline{\lambda}_2$, $\lambda_{7a} := \lambda_7 - \underline{\lambda}_1 - \underline{\lambda}_2$, and $\underline{\lambda}_8 := \lambda_9 - \underline{\lambda}_4$. Now $\underline{\lambda}_5$ and $\underline{\lambda}_8$ are Brauer atoms, thus irreducible.

Table 14.2: Some characters for b_{19}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(16)$	128	$\Lambda[\![5^2, 4:2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}/2$	18672000
2	$\lambda[\![6,5,4]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	768	$\Lambda[\![6,5,4]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	9536000
3	$\lambda[\![7,5,3]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	21760	$\lambda(16) \otimes \Lambda[12,3,1] \|_{b_{19}}/2$	1760000
4	$\lambda[\![6,4,3,2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	25088	$\Lambda[\![6,4,3,2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	12608000
5	$\lambda(16) \otimes \lambda[9,7] \|_{b_{19}}$	48256	$\Sigma[\![5,4,3,2,+]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	12352000
6	$\lambda[\![8,5,2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	95744	$\Lambda[\![8,6,1,+]\!]\uparrow^{\tilde{A}_{16}}$	2496000
7	$\lambda\langle\!\langle 11,5,-\rangle\!\rangle\|_{5'}$	104832	$\Lambda[\![9,5,1]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}/3$	2496000
8	$\lambda\langle\!\langle 10,6,+\rangle\!\rangle\|_{5'}$	128128	$\Lambda[\![8,3,2,+:2,1]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}/2$	4960000
9	$\lambda\langle\!\langle 10,3,2,1,-\rangle\!\rangle\|_{5'}$	150528	$\Lambda[\![6,5,3,1,+]\!]\uparrow^{\tilde{A}_{16}}$	4160000
10	$\lambda[\![5^2,3,2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	160512	$\Lambda[\![7,4,2,+:3,-]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}/2$	4576000
11	$\lambda\langle\!\langle 7,6,3\rangle\!\rangle\|_{5'}$	630784	$\lambda(16) \otimes \Lambda[15,1] \|_{b_{19}}$	36144000
12	$\lambda\langle\!\langle 6,5,3,2,+\rangle\!\rangle\|_{5'}$	209664	$\Lambda[\![5^2,3,1,+:2]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	26496000
13	$\lambda(16) \otimes \lambda[11,5] \|_{b_{19}}$	208000		
14	$\lambda\langle\!\langle 11,3,2\rangle\!\rangle\|_{5'}$	221184		
15	$\lambda[\![7,4,3,1]\!]\uparrow^{\tilde{A}_{16}} \|_{b_{19}}$	1042944		

Table 14.3: Mutual scalar products (b_{19})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
λ_1	1	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.
λ_3	.	.	1	.	.	.	.	.	.	.
λ_4	.	.	.	1	.	.	.	.	.	.
λ_{5a}	1	1	.	.	1	.	.	.	.	.
λ_6	.	.	.	.	.	1	.	.	.	.
λ_7	3	1	.	.	1	.	1	.	.	.
λ_9	.	.	.	1	.	.	.	1	.	.
λ_{10}	.	.	.	.	.	.	.	.	1	.
λ_{11}	2	2	1	2	2	.	.	.	.	1

Moreover, $\lambda_{15} = -4\underline{\lambda}_1 - 4\underline{\lambda}_2 - 2\underline{\lambda}_3 - 3\underline{\lambda}_4 - 4\underline{\lambda}_5 + 2\lambda_{11}$ gives $\underline{\lambda}_{10} := \lambda_{10} - 2\underline{\lambda}_1 - 2\underline{\lambda}_2 - \underline{\lambda}_3 - 2\underline{\lambda}_4 - 2\underline{\lambda}_5$ which is irreducible. Then $\Lambda_{11} = \Lambda_1 + 2\underline{\Lambda}_6 - 2\underline{\Lambda}_7 + 2\underline{\Lambda}_9 + 2\underline{\Lambda}_{10}$ and $\Lambda_{12} = \Lambda_5 - \underline{\Lambda}_7 + 4\underline{\Lambda}_9$ imply that $\underline{\lambda}_7 := \lambda_{7a}$ is irreducible. The decomposition matrix of b_{19} is Table 14.10.

14.3 The block B_{30}

We have basic sets $\{\sigma_1, \ldots, \sigma_{20}\}$ and $\{\Sigma_1, \ldots, \Sigma_{20}\}$, see Tables 14.4 and 14.5.

Table 14.4: Some characters for B_{30}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(16)$	128	$\sigma(16) \otimes \Sigma[4, 3^4] \mid_{B_{30}}$	23664000
2	$\sigma(16)^a$	128	$\sigma(16) \otimes \Sigma[13, 1^3] \mid_{B_{30}}$	23664000
3	$\sigma[\![6,5,4,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	768	$\Sigma[\![6,5,4,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	9536000
4	$\sigma[\![6,5,4,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	768	$\Sigma[\![6,5,3:2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	16192000
5	$\sigma[\![7,5,3,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	21760	$\Sigma[\![6,5,4,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	9536000
6	$\sigma[\![7,5,3,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	21760	$\Sigma[\![7,5,3,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	6336000
7	$\sigma[\![6,4,3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	25088	$\Sigma[\![7,5,3,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	6336000
8	$\sigma[\![6,4,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	25088	$\Sigma[\![6,4,3:2,1,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	12608000
9	$\underline{\lambda}_5(b_{19})\uparrow^{\tilde{S}_{16}}$	48128	$\Sigma[\![6,4,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	12608000
10	$\sigma[\![5,4,1:5,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	339712	$\Sigma[\![5^2,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	14016000
11	$\sigma[\![8,5,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	95744	$\Sigma[\![5^2,3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	14016000
12	$\sigma[\![8,5,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	95744	$\Sigma[\![8,5,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	9952000
13	$\sigma[\![5^2,1:4,1,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	106752	$\Sigma[\![8,5,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	9952000
14	$\sigma[\![5^2,1:4,1,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	106752	$\underline{\Lambda}_7(b_{19})\uparrow^{\tilde{S}_{16}}$	4992000
15	$\sigma[\![5^2,3,2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	160512	$\Sigma[\![9,5,1,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	7488000
16	$\sigma[\![5^2,3,2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	160512	$\Sigma[\![8,5,1:2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	17440000
17	$\sigma\langle\!\langle 11,3,2,- \rangle\!\rangle\mid_{5'}$	221184	$\underline{\Lambda}_9(b_{19}\uparrow^{\tilde{S}_{16}}$	8320000
18	$\sigma\langle\!\langle 11,3,2,+ \rangle\!\rangle\mid_{5'}$	221184	$\underline{\Lambda}_8(b_{19}\uparrow^{\tilde{S}_{16}}$	9920000
19	$\sigma\langle\!\langle 7,6,3,- \rangle\!\rangle\mid_{5'}$	630784	$\underline{\Lambda}_{10}(b_{19}\uparrow^{\tilde{S}_{16}}$	9152000
20	$\sigma\langle\!\langle 7,6,3,+ \rangle\!\rangle\mid_{5'}$	630784	$\Sigma[\![7,5,1,+:3]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	29056000
21	$\sigma[\![5^2,3,1:2,+]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	1633280		
22	$\sigma[\![5^2,3,1:2,-]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{30}}$	1633280		

From $\underline{\lambda}_i(b_{19})\uparrow^{\tilde{S}_{16}} = \sigma_{2i-1} + \sigma_{2i}$ for $i \in \{1,2,3,4,6\}$ and $\underline{\lambda}_9(b_{19})\uparrow^{\tilde{S}_{16}} = \sigma_{15} + \sigma_{16}$ we see that $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1, \ldots, 8, 11, 12, 15, 16\}$. Furthermore, we have that $\underline{\lambda}_5(b_{19})\uparrow^{\tilde{S}_{16}} = \sigma_9 = \Sigma_{10}^* + \Sigma_{11}^*$. This identifies the Brauer atoms $\underline{\sigma}_9 := \Sigma_{10}^*$ and $\underline{\sigma}_{10} := \Sigma_{11}^*$

Table 14.5: Mutual scalar products (B_{30})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}	Σ_{19}	Σ_{20}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_4	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_5	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	2
σ_6	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	2
σ_7	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
σ_8	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
σ_9	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
σ_{10}	6	6	2	2	2	.	.	.	.	1	.	1	2	3	5	1	.	.	.	1
σ_{11}	1	1	.	1	.	.	.	.	.	.	.	1	.	.	.	2	.	.	.	2
σ_{12}	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	2	.	.	.	2
σ_{13}	3	3	1	2	2	.	.	.	.	.	.	.	1	1	2	.	.	.	.	.
σ_{14}	3	3	2	1	1	.	.	.	.	.	.	1	.	1	1	1	.	.	.	1
σ_{15}	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.
σ_{16}	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.
σ_{17}	1	1	.	1	.	.	.	.	.	.	.	1	1	.	.	2	.	1	.	2
σ_{18}	1	1	.	.	.	.	.	.	.	.	.	1	1	.	.	3	.	1	.	3
σ_{19}	1	1	1	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	1	3
σ_{20}	1	1	1	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	1	2

as irreducible Brauer characters. Consider the relation

$$\underline{\lambda}_7(b_{19})\uparrow^{\tilde{S}_{16}} = -4\underline{\sigma}_1 - 4\underline{\sigma}_2 - 3\underline{\sigma}_3 - 3\underline{\sigma}_4 + \sigma_{13} + \sigma_{14}.$$

It is clear by Table 14.5 that $\underline{\sigma}_3$ is contained in σ_{13} once and in σ_{14} twice, whereas $\underline{\sigma}_4$ is contained in σ_{14} once and in σ_{13} twice. As $\underline{\sigma}_1$ and $\underline{\sigma}_2$ are fixed by complex conjugation, but $\sigma_{13}^{\circledast} = \sigma_{14}$, we can deduce that $2(\underline{\sigma}_1 + \underline{\sigma}_2)$ is contained in both, σ_{13} and σ_{14}. Hence $\underline{\sigma}_{13} := \sigma_{13} - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_3 - 2\underline{\sigma}_4$ and $\underline{\sigma}_{14} := \sigma_{14} - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 2\underline{\sigma}_3 - 1\underline{\sigma}_4$ are irreducible. Then $\underline{\lambda}_8(b_{19})\uparrow^{\tilde{S}_{16}} = -\underline{\sigma}_{11} - \underline{\sigma}_{12} + \sigma_{17} + \sigma_{18}$ and Table 14.5 imply that $\underline{\sigma}_{17} := \sigma_{17} - \underline{\sigma}_{11}$ and $\underline{\sigma}_{18} := \sigma_{18} - \underline{\sigma}_{12}$ are irreducible too. Similarly,

$$\underline{\lambda}_{10}(b_{19})\uparrow^{\tilde{S}_{16}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 2\underline{\sigma}_3 - 2\underline{\sigma}_4 - \underline{\sigma}_5 - \underline{\sigma}_6 - 2\underline{\sigma}_7 - 2\underline{\sigma}_8 - 2\sigma_9 + \sigma_{19} + \sigma_{20}$$

and the corresponding decompositions in terms of the Brauer atoms yield

$$\sigma_{ia} := \sigma_i - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_3 - \underline{\sigma}_4 - \underline{\sigma}_7 - \underline{\sigma}_8 - \underline{\sigma}_9 - \underline{\sigma}_{10}$$

for $i \in \{19, 20\}$ (recall $\sigma_9 = \underline{\sigma}_9 + \underline{\sigma}_{10}$). It remains to decide which element of $\{\underline{\sigma}_5, \underline{\sigma}_6\}$ has to be subtracted from σ_{19a} (or σ_{20a}). This is clear by

$$\sigma_{21} = 2\underline{\sigma}_1 + 2\underline{\sigma}_2 + \underline{\sigma}_3 + 2\underline{\sigma}_4 - \underline{\sigma}_6 + 3\underline{\sigma}_7 + 3\underline{\sigma}_8 + 4\underline{\sigma}_9 + 3\underline{\sigma}_{10} + 3\underline{\sigma}_{15} + 2\underline{\sigma}_{16} + \sigma_{19a}.$$

(The decomposition of σ_{22} shows that $\underline{\sigma}_5$ is contained in σ_{20a}, but this is obvious now anyway by taking associates.) Thus $\underline{\sigma}_{19} := \sigma_{19a} - \underline{\sigma}_6$ and $\underline{\sigma}_{20} := \sigma_{20a} - \underline{\sigma}_5$ complete $\mathrm{IBr}(B_{30})$. The decomposition matrix of B_{30} is Table 14.11.

14.4 The block B_{31}

The Brauer characters $\underline{\sigma}_i$ and the projective characters $\underline{\Sigma}_j$ that are defined by Table 14.6 satisfy $\langle \underline{\sigma}_i, \underline{\Sigma}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, 5\}$. The decomposition matrix of B_{31} is Table 14.12.

14.5 The block b_{20}

We begin with a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1, \ldots, 10\}$ defined by Table 14.7.

Table 14.6: Some characters for B_{31}

i	$\underline{\sigma}_i$	$\deg \underline{\sigma}_i$	$\underline{\Sigma}_i$	$\deg \underline{\Sigma}_i$
1	$\sigma\langle\langle 14, 2\rangle\rangle\vert_{5'}$	11520	$\Sigma[\![7, 5, 3, -]\!]\uparrow^{\tilde{S}_{16}}\vert_{B_{31}}$	3456000
2	$\sigma(16) \otimes \sigma[9, 7]\vert_{B_{31}}$	78080	$\sigma(16) \otimes \Sigma[5^2, 2^3]\vert_{B_{31}}$	4576000
3	$\underline{\sigma}_3(B_{30}) \otimes \sigma[11, 5]\vert_{B_{31}}$	104960	$\sigma(16) \otimes \Sigma[9, 5, 1^2]\vert_{B_{31}}$	4160000
4	$\underline{\sigma}_7(B_{30}) \otimes \sigma[15, 1]$	376320	$\Sigma[\![7, 4, 3, 1, +]\!]\uparrow^{\tilde{S}_{16}}\vert_{B_{31}}$	4864000
5	$\sigma[\![8, 4, 2, 1, -]\!]\uparrow^{\tilde{S}16}\vert_{B_{31}}$	627200	$\Sigma[\![9, 4 : 2, 1]\!]\uparrow^{\tilde{S}_{16}}\vert_{B_{31}}$	4864000

Table 14.7: Some characters for b_{20}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\sigma[\![5^2, 4, -]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	5760	$\lambda\langle\langle 12, 4, +\rangle\rangle \otimes \lambda\langle 4, 2^4, 1^4\rangle\vert_{b_{20}}$	1441680000
2	$\sigma[\![5^2, 4, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	5760	$\underline{\Sigma}_1(B_{31})\downarrow_{\tilde{A}_{16}}$	3456000
3	$\lambda\langle\langle 12, 4, -\rangle\rangle\vert_{5'}$	58240	$\Sigma[\![5^2, 4, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	20688000
4	$\lambda\langle\langle 12, 4, +\rangle\rangle\vert_{5'}$	58240	$\underline{\Sigma}_3(B_{31})\downarrow_{\tilde{A}_{16}}$	4160000
5	$\lambda\langle\langle 9, 7, +\rangle\rangle\vert_{5'}$	91520	$\lambda(16) \otimes \Lambda[5^2, 2^3, -]\vert_{b_{20}}$	2288000
6	$\underline{\sigma}_2(B_{31})\downarrow_{\tilde{A}_{16}}$	78080	$\lambda(16) \otimes \Lambda[5^2, 2^3, +]\vert_{b_{20}}$	2288000
7	$\sigma[\![7, 4, 3, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	188160	$\Sigma[\![7, 4, 3, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	4864000
8	$\sigma[\![7, 4, 3, -]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	188160	$\Lambda[\![7, 4, 2, + : 3, -]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	4864000
9	$\sigma[\![9, 4, 1, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	418560	$\Lambda[\![9, 4, 2, +]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	2432000
10	$\sigma[\![9, 4, 1, -]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	418560	$\Lambda[\![9, 4, 2, -]\!]\uparrow^{\tilde{A}_{16}}\vert_{b_{20}}$	2432000

In view of the first column of Table 14.8 we already get an impression of the problem that too many pairs of conjugate characters are “glued together”, namely $\lambda_6 = \underline{\sigma}_2(B_{31}){\downarrow_{\tilde{A}_{16}}}$, $\Lambda_2 = \underline{\Sigma}_1(B_{31}){\downarrow_{\tilde{A}_{16}}}$, and $\Lambda_4 = \underline{\Sigma}_3(B_{31}){\downarrow_{\tilde{A}_{16}}}$. However, $\underline{\sigma}_1(B_{31}){\downarrow_{\tilde{A}_{16}}} = \lambda_1 + \lambda_2$ and $\underline{\sigma}_4(B_{31}){\downarrow_{\tilde{A}_{16}}} = \lambda_7 + \lambda_8$ show that $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1, 2, 7, 8\}$.

Table 14.8: Mutual scalar products (b_{20})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
λ_1	18	1	.	.	.	.	.	.	.	.
λ_2	18	1	1	.	.	.	.	.	.	.
λ_3	97	1	1	1	.	.	.	.	.	.
λ_4	96	1	1	1	.	.	.	.	.	.
λ_5	79	.	1	1	1	.	.	.	.	.
λ_6	1	.	1	.	1	1	.	.	.	.
λ_7	29	.	.	.	.	.	1	.	.	.
λ_8	29	.	.	.	.	.	.	1	.	.
λ_9	344	.	4	2	.	.	1	.	1	.
λ_{10}	344	.	4	2	.	.	.	1	.	1

We have $\underline{\sigma}_3(B_{31}){\downarrow_{\tilde{A}_{16}}} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \lambda_3 + \lambda_4$ and $\underline{\sigma}_5(B_{31}){\downarrow_{\tilde{A}_{16}}} = 2\underline{\lambda}_1 + 2\underline{\lambda}_2 - 2\lambda_3 - 2\lambda_4 + \lambda_9 + \lambda_{10}$. These equations imply that irreducible constituents $\underline{\lambda}_9$ and $\underline{\lambda}_{10}$ of λ_9 and λ_{10}, respectively, can be expressed as

$$\underline{\lambda}_9 = \lambda_9 + x_1\underline{\lambda}_1 + x_2\underline{\lambda}_2 - x_3\lambda_3 - x_4\lambda_4,$$
$$\underline{\lambda}_{10} = \lambda_{10} + y_1\underline{\lambda}_1 + y_2\underline{\lambda}_2 - y_3\lambda_3 - y_4\lambda_4$$

for suitable integers x_i and y_i such that $0 \leqslant x_i, y_i \leqslant 2$ and $x_i + y_i = 2$ for $i \in \{1, \ldots, 4\}$. As $\underline{\lambda}_1^{\circledast} = \underline{\lambda}_2$, $\lambda_3^{\circledast} = \lambda_4$ and $\lambda_i^{\circledast} = \lambda_i$ for $i \in \{9, 10\}$, it follows that $x_1 = x_2$, $x_3 = x_4$, $y_1 = y_2$, and $y_3 = y_4$. In particular, $\deg \underline{\lambda}_9 = \deg \underline{\lambda}_{10}$ implies $x_i = y_i = 1$ for all $i \in \{1, \ldots, 4\}$. Hence we have determined $\underline{\lambda}_9$ and $\underline{\lambda}_{10}$.

The projective characters $\underline{\Lambda}_9 := \Lambda_9$ and $\underline{\Lambda}_{10} := \Lambda_{10}$ are indecomposable, see Table 14.8. Then $\underline{\Sigma}_4(B_{31}){\downarrow_{\tilde{A}_{16}}} = \Lambda_7 + \Lambda_8 - \underline{\Lambda}_9 - \underline{\Lambda}_{10}$ gives $\underline{\Lambda}_7 := \Lambda_7 - \underline{\Lambda}_9$ and $\underline{\Lambda}_8 := \Lambda_8 - \underline{\Lambda}_{10}$ which are indecomposable too. Moreover, $\underline{\Lambda}_5 := \Lambda_5$ and $\underline{\Lambda}_6 := \Lambda_6$ are indecomposable since $\underline{\Sigma}_2(B_{31}){\downarrow_{\tilde{A}_{16}}} = \Lambda_5 + \Lambda_6$. By the definition of Λ_2 and Λ_4 we know that each of them is the sum of two conjugate, projective indecomposable characters but it is not clear how to split them. We can improve the basic set of projective characters but this will not give more information on the missing irreducible Brauer characters of b_{20} (therefore, the hasty reader may want to skip the next paragraph).

As $\underline{\lambda}_i$ corresponds to $\underline{\Lambda}_i$ for $i \in \{7, \dots, 10\}$, we get $\Lambda_{1a} := \Lambda_1 - 29\underline{\Lambda}_7 - 29\underline{\Lambda}_8 - 187\underline{\Lambda}_9 - 187\underline{\Lambda}_{10}$ instead of Λ_1, and $\Lambda_{3a} := \Lambda_3 - 3\underline{\Lambda}_9 - 3\underline{\Lambda}_{10}$ improves Λ_3. Now we use the interplay between $\Lambda_2 = \underline{\Sigma}_1(B_{31})\downarrow_{\tilde{A}_{16}}$ and $\Lambda_{1a}\uparrow^{\tilde{S}_{16}} = 36\underline{\Sigma}_1(B_{31}) + \underline{\Sigma}_2(B_{31}) + 157\underline{\Sigma}_3(B_{31})$. As $\deg \underline{\lambda}_1^* = \deg \underline{\lambda}_2^* < 0$ and $\deg \lambda_3^* = \deg \lambda_4^* > 0$, we know that the indecomposable constituents of Λ_2, say $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$, are either $\underline{\lambda}_1^* + \lambda_3^*$ and $\underline{\lambda}_2^* + \lambda_4^*$, or $\underline{\lambda}_1^* + \lambda_4^*$ and $\underline{\lambda}_2^* + \lambda_3^*$. In both cases, each constituent can be contained at most 18 times in Λ_{1a}, see Table 14.8. Conversely, Λ_{1a} contains $x_1\underline{\Lambda}_1 + x_2\underline{\Lambda}_2$ where $x_1, x_2 \in \mathbb{Z}^+$ such that $x_1 + x_2 = 36$. Thus $x_1 = x_2 = 18$ and we can improve Λ_{1a} by $\Lambda_{1b} := \Lambda_{1a} - 18\Lambda_2$. Similarly, as $\deg \lambda_5^* = 0$, the indecomposable constituents of Λ_4, say $\underline{\Lambda}_3$ and $\underline{\Lambda}_4$, are either λ_3^* and $\lambda_4^* + \lambda_5^*$, or $\lambda_3^* + \lambda_5^*$ and λ_4^*. In the first case, since λ_3^* can be contained in Λ_{1b} with multiplicity at most 79, $\lambda_4^* + \lambda_5^*$ must be contained in Λ_{1b} with multiplicity at least $157 - 79 = 78$. As the multiplicity of $\lambda_4^* + \lambda_5^*$ in Λ_{1b} is at most 78, it follows that Λ_4 can be subtracted 78 times from Λ_{1b}. The second case is completely analogous. Thus $\Lambda_{1c} := \Lambda_{1b} - 78\Lambda_4$ improves Λ_{1b}.

Table 14.9: Mutual scalar products (b_{20})

$\langle\cdot,\cdot\rangle$	Λ_2	Λ_{3a}	Λ_4	Λ_{1c}	$\underline{\Lambda}_5$	$\underline{\Lambda}_6$	$\underline{\Lambda}_7$	$\underline{\Lambda}_8$	$\underline{\Lambda}_9$	$\underline{\Lambda}_{10}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	1	1	.	.	.	.	.	.	.	.
λ_4	1	1	1	.	.	.	.	.	.	.
λ_3	1	1	1	1	.	.	.	.	.	.
λ_5	.	1	1	1	1	.	.	.	.	.
λ_6	.	1	.	1	1	1	.	.	.	.

Recall that $\underline{\sigma}_3(B_{31})\downarrow_{\tilde{A}_{16}} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \lambda_3 + \lambda_4$. We use condensation and the trace formula to prove that $\underline{\lambda}_1$ is contained in λ_3, and $\underline{\lambda}_2$ is contained in λ_4. Let $128a^-$ be a basic spin module of $\tilde{A}_{16}$ over $F := GF(25)$. Let $820a^+$ be the $F\tilde{A}_{16}$-module with Brauer character $\lambda[12,4]$ constructed by chopping the sequence $(15a^+ \otimes 15a^+, 15a^+ \otimes 103a^+, 15a^+ \otimes 440a^+)$ where $15a^+$ denotes the reduced permutation module of S_{16} restricted to A_{16}. Since $\lambda(16) \otimes \lambda[12,4] = \underline{\sigma}_3(B_{31})\downarrow_{\tilde{A}_{16}}$, the tensor product $V := 128a^- \otimes 820a^+$ suits us well. We choose the condensation subgroup $K := \langle t_{(9,1^7)}, zt_{10}t_{11}, zt_{14}t_{15}\rangle \cong 9 \times 3^2$ and we apply the trace formula with respect to $x := (1,4,7,10,13)(2,5,8,12,15,3,6,9,16,11,14) \in A_{16}$.

Let $\phi := \lambda_3 - \underline{\lambda}_1$ with $\phi^c = \lambda_4 - \underline{\lambda}_2$, and $\psi := \lambda_3 - \underline{\lambda}_2$ with $\psi^c = \lambda_4 - \underline{\lambda}_1$. Then

$$|K|^{-1} \sum_{k \in K} \overline{\breve{\varphi}(\tilde{x}k)} = \begin{cases} 0_F & \text{if } \varphi = \phi, \\ 0_F & \text{if } \varphi = \phi^c, \\ (\zeta_{25})^{15} & \text{if } \varphi = \psi, \\ (\zeta_{25})^{3} & \text{if } \varphi = \psi^c. \end{cases}$$

Then $\langle \underline{\lambda}_i \downarrow_K, 1_K \rangle = 8$ for $i \in \{1,2\}$ and $\langle \lambda_i \downarrow_K, 1_K \rangle = 312$ for $i \in \{3,4\}$. We define $a := zt_1t_2$, $b := t_1t_{15}t_{14} \cdots t_1$, $c := ((ab)^2(bab)^2abab^3)^2ab^2$, and $d := c(ab)^3b$. Put $e := e_K$ and

$$C(\tilde{x}) := \langle ece, ede, e\tilde{x}e \rangle \leqslant eF\tilde{A}_{16}e =: H.$$

The H-module Ve has dimension 608 and two distinct composition factors of equal dimension 304. In fact, Ve already has two distinct $C(\tilde{x})$-composition factors 304a and 304b with $\mathrm{tr}_{304a}(e\tilde{x}e) = 0_F = \mathrm{tr}_{304b}(e\tilde{x}e)$. Thus $\phi, \phi^c \in \mathrm{IBr}(b_{20})$.
Then we can split $\Lambda_2 = \underline{\Sigma}_1(B_{31})\downarrow_{\tilde{A}_{16}} = \underline{\Lambda}_1^* + \underline{\Lambda}_2^*$. Exchanging Λ_2 and Λ_{3a} with $\underline{\Lambda}_1^*$ and $\underline{\Lambda}_2^*$ yields $\lambda_6 = \Lambda_{1c}^* + \underline{\Lambda}_5^* + \underline{\Lambda}_6^*$ but this still allows two distinct possibilities of how to split $\lambda_6 = \underline{\sigma}_2(B_{31})\downarrow_{\tilde{A}_{16}}$. Let $\eta := \Lambda_{1c}^* + \underline{\Lambda}_5^*$ with $\eta^c = \underline{\Lambda}_6^*$, and $\theta := \Lambda_{1c}^* + \underline{\Lambda}_6^*$ with $\theta^c = \underline{\Lambda}_5^*$. We condense $W := 128a^- \otimes 6435a^+$ where $6435a^+$ affords $\lambda[9, 2^3, 1]$. Let ν be the vector $(2, 4, 0, \dots, 0) \cdot 1_F \in F^{128}$. We construct $6435a^+$ by spinning up $\nu \otimes \nu$ with respect to $128a^- \otimes 128a^-$. We have $\langle \lambda_6 \downarrow_K, 1_K \rangle = 1280$, hence we are on the outlook for two distinct H-composition factors of We of dimension 640. To distinguish the Brauer characters in question, we find $y := (1, 8, 15, 7, 14, 6, 13, 5, 12, 4, 11, 3, 10, 2, 9) \in A_{16}$ with

$$|K|^{-1} \sum_{k \in K} \overline{\breve{\varphi}(\tilde{y}k)} = \begin{cases} (\zeta_{25})^{23} & \text{if } \varphi = \eta, \\ (\zeta_{25})^{19} & \text{if } \varphi = \eta^c, \\ (\zeta_{25})^{14} & \text{if } \varphi = \theta, \\ (\zeta_{25})^{22} & \text{if } \varphi = \theta^c. \end{cases}$$

Then We has two distinct $C(\tilde{y})$-composition factors 640a and 640b. It is easy to see that these are the ones that correspond to $\{\eta, \eta^c\}$ or $\{\theta, \theta^c\}$. We conclude that $\eta, \eta^c \in \mathrm{IBr}(b_{20})$ since $\mathrm{tr}_{640a}(e\tilde{y}e) = (\zeta_{25})^{19}$ and $\mathrm{tr}_{640b}(e\tilde{y}e) = (\zeta_{25})^{23}$. The decomposition matrix of b_{20} is Table 14.13.

Table 14.10: 5-modular decomposition matrix of block b_{19} of $\tilde{A}_{16}$
(defect 3, bar core (1), weight 3, content (7, 6, 3))

			$[\![5^2,3,2,1]\!]$ 24064	$[\![5^3,1]\!]$ 128	$[\![6,4,3,2,1]\!]$ 25088	$[\![6,5,3,2]\!]$ 160512	$[\![6,5,4,1]\!]$ 768	$[\![7,5,3,1]\!]$ 508928	$[\![7,6,3]\!]$ 21760	$[\![8,5,2,1]\!]$ 125440	$[\![8,6,2]\!]$ 95744	$[\![9,6,1]\!]$ 103936
$\star$	128	$\langle\!\langle 16\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
$\star$	896	$\langle\!\langle 15,1,+\rangle\!\rangle$	.	1	.	.	1	.	.	.	.	.
	896	$\langle\!\langle 15,1,-\rangle\!\rangle$	.	1	.	.	1	.	.	.	.	.
$\star$	22528	$\langle\!\langle 13,2,1\rangle\!\rangle$	.	.	.	.	1	.	1	.	.	.
$\star$	118272	$\langle\!\langle 12,3,1\rangle\!\rangle$	.	.	.	.	1	.	1	.	1	.
$\star$	104832	$\langle\!\langle 11,5,+\rangle\!\rangle$	.	1	.	.	1	.	.	.	.	1
	104832	$\langle\!\langle 11,5,-\rangle\!\rangle$	.	1	.	.	1	.	.	.	.	1
	326144	$\langle\!\langle 11,4,1\rangle\!\rangle$	.	2	.	.	1	.	.	1	1	1
	221184	$\langle\!\langle 11,3,2\rangle\!\rangle$	.	.	.	.	.	.	.	1	1	.
$\star$	128128	$\langle\!\langle 10,6,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	128128	$\langle\!\langle 10,6,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	559104	$\langle\!\langle 10,5,1\rangle\!\rangle$	2	4	2	.	2	.	.	2	.	2
$\star$	150528	$\langle\!\langle 10,3,2,1,+\rangle\!\rangle$	.	.	1	.	.	.	.	1	.	.
	150528	$\langle\!\langle 10,3,2,1,-\rangle\!\rangle$	.	.	1	.	.	.	.	1	.	.
	585728	$\langle\!\langle 9,6,1\rangle\!\rangle$	2	2	2	1	2	.	.	1	1	1
	256256	$\langle\!\langle 8,7,1\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.
	988416	$\langle\!\langle 8,6,2\rangle\!\rangle$	1	2	2	1	2	1	1	1	1	.
	709632	$\langle\!\langle 8,5,2,1,+\rangle\!\rangle$	1	2	2	.	1	1	.	1	.	.
	709632	$\langle\!\langle 8,5,2,1,-\rangle\!\rangle$	1	2	2	.	1	1	.	1	.	.
$\star$	630784	$\langle\!\langle 7,6,3\rangle\!\rangle$	2	2	2	.	2	1	1	.	.	.
	768768	$\langle\!\langle 7,5,3,1,+\rangle\!\rangle$	2	2	2	1	1	1	.	.	.	.
	768768	$\langle\!\langle 7,5,3,1,-\rangle\!\rangle$	2	2	2	1	1	1	.	.	.	.
$\star$	186368	$\langle\!\langle 6,5,4,1,+\rangle\!\rangle$	.	.	1	1	1	.	.	.	.	.
	186368	$\langle\!\langle 6,5,4,1,-\rangle\!\rangle$	.	.	1	1	1	.	.	.	.	.
	209664	$\langle\!\langle 6,5,3,2,+\rangle\!\rangle$	1	.	1	1	.	.	.	.	.	.
	209664	$\langle\!\langle 6,5,3,2,-\rangle\!\rangle$	1	.	1	1	.	.	.	.	.	.
$\star$	73216	$\langle\!\langle 6,4,3,2,1\rangle\!\rangle$	2	.	1	.	.	.	.	.	.	.

Table 14.11: 5-modular decomposition matrix of block B_{30} of $\tilde{S}_{16}$
(defect 3, bar core (1), weight 3, content $(7,6,3)$)

			$[\![5^2,3,2,1,+]\!]$ 24064	$[\![5^2,3,2,1,-]\!]$ 24064	$[\![5^3,1,+]\!]$ 128	$[\![5^3,1,-]\!]$ 128	$[\![6,4,3,2,1,+]\!]$ 25088	$[\![6,4,3,2,1,-]\!]$ 25088	$[\![6,5,3,2,+]\!]$ 160512	$[\![6,5,3,2,-]\!]$ 160512	$[\![6,5,4,1,+]\!]$ 768	$[\![6,5,4,1,-]\!]$ 768	$[\![7,5,3,1,+]\!]$ 508928	$[\![7,5,3,1,-]\!]$ 508928	$[\![7,6,3,+]\!]$ 21760	$[\![7,6,3,-]\!]$ 21760	$[\![8,5,2,1,+]\!]$ 125440	$[\![8,5,2,1,-]\!]$ 125440	$[\![8,6,2,+]\!]$ 95744	$[\![8,6,2,-]\!]$ 95744	$[\![9,6,1,+]\!]$ 103936	$[\![9,6,1,-]\!]$ 103936
⋆	128	$\langle\!\langle 16,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	128	$\langle\!\langle 16,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1792	$\langle\!\langle 15,1\rangle\!\rangle$	.	.	1	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
⋆	22528	$\langle\!\langle 13,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
⋆	22528	$\langle\!\langle 13,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.
⋆	118272	$\langle\!\langle 12,3,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
⋆	118272	$\langle\!\langle 12,3,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.
⋆	209664	$\langle\!\langle 11,5\rangle\!\rangle$	.	.	1	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1
⋆	326144	$\langle\!\langle 11,4,1,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1	.	1	1	.
	326144	$\langle\!\langle 11,4,1,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	1
⋆	221184	$\langle\!\langle 11,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.
⋆	221184	$\langle\!\langle 11,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.
⋆	256256	$\langle\!\langle 10,6\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
	559104	$\langle\!\langle 10,5,1,+\rangle\!\rangle$	1	1	2	2	1	1	.	.	1	1	.	.	.	.	1	1	.	.	1	1
	559104	$\langle\!\langle 10,5,1,-\rangle\!\rangle$	1	1	2	2	1	1	.	.	1	1	.	.	.	.	1	1	.	.	1	1
	301056	$\langle\!\langle 10,3,2,1\rangle\!\rangle$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	.	.
⋆	585728	$\langle\!\langle 9,6,1,+\rangle\!\rangle$	1	1	1	1	1	1	1	.	1	1	.	.	.	.	1	.	1	.	.	1
	585728	$\langle\!\langle 9,6,1,-\rangle\!\rangle$	1	1	1	1	1	1	.	1	1	1	.	.	.	.	.	1	.	1	1	.
⋆	256256	$\langle\!\langle 8,7,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.
⋆	256256	$\langle\!\langle 8,7,1,-\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.
⋆	988416	$\langle\!\langle 8,6,2,+\rangle\!\rangle$	.	1	1	1	1	1	1	.	1	1	1	.	.	1	1	.	1	.	.	.
	988416	$\langle\!\langle 8,6,2,-\rangle\!\rangle$	1	.	1	1	1	1	.	1	1	1	.	1	1	.	.	1	.	1	.	.
	1419264	$\langle\!\langle 8,5,2,1\rangle\!\rangle$	1	1	2	2	2	2	.	.	1	1	1	1	.	.	1	1	.	.	.	.
⋆	630784	$\langle\!\langle 7,6,3,+\rangle\!\rangle$	1	1	1	1	1	1	.	.	1	1	.	1	1	.	.	.	.	.	.	.
⋆	630784	$\langle\!\langle 7,6,3,-\rangle\!\rangle$	1	1	1	1	1	1	.	.	1	1	1	.	.	1	.	.	.	.	.	.
	1537536	$\langle\!\langle 7,5,3,1\rangle\!\rangle$	2	2	2	2	2	2	1	1	1	1	1	1	.	.	.	.	.	.	.	.
	372736	$\langle\!\langle 6,5,4,1\rangle\!\rangle$	.	.	.	.	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.
	419328	$\langle\!\langle 6,5,3,2\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	73216	$\langle\!\langle 6,4,3,2,1,+\rangle\!\rangle$	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	73216	$\langle\!\langle 6,4,3,2,1,-\rangle\!\rangle$	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 14.12: 5-modular decomposition matrix of block B_{31} of $\tilde{S}_{16}$ (defect 2, bar core $(4,2)$, weight 2, content $(6,7,3)$)

			$[\![5^2,4,2]\!]$ 78080	$[\![7,4,3,2]\!]$ 376320	$[\![7,5,4]\!]$ 11520	$[\![9,4,2,1]\!]$ 627200	$[\![9,5,2]\!]$ 104960
$\star$	11520	$\langle\!\langle 14,2\rangle\!\rangle$	.	.	1	.	.
$\star$	116480	$\langle\!\langle 12,4\rangle\!\rangle$	.	.	1	.	1
$\star$	732160	$\langle\!\langle 10,4,2,+\rangle\!\rangle$	.	.	.	1	1
	732160	$\langle\!\langle 10,4,2,-\rangle\!\rangle$	.	.	.	1	1
$\star$	183040	$\langle\!\langle 9,7\rangle\!\rangle$	1	.	.	.	1
	1198080	$\langle\!\langle 9,5,2,+\rangle\!\rangle$	1	1	1	1	1
	1198080	$\langle\!\langle 9,5,2,-\rangle\!\rangle$	1	1	1	1	1
	1003520	$\langle\!\langle 9,4,2,1\rangle\!\rangle$	.	1	.	1	.
$\star$	465920	$\langle\!\langle 7,5,4,+\rangle\!\rangle$	1	1	1	.	.
	465920	$\langle\!\langle 7,5,4,-\rangle\!\rangle$	1	1	1	.	.
	532480	$\langle\!\langle 7,4,3,2\rangle\!\rangle$	2	1	.	.	.

Table 14.13: 5-modular decomposition matrix of block $\mathfrak{b}_{20}$ of $\tilde{A}_{16}$ (defect 2, bar core $(4,2)$, weight 2, content $(6,7,3)$)

			$[\![5^2,4,2,+]\!]$ 39040	$[\![5^2,4,2,-]\!]$ 39040	$[\![7,4,3,2,+]\!]$ 188160	$[\![7,4,3,2,-]\!]$ 188160	$[\![7,5,4,+]\!]$ 5760	$[\![7,5,4,-]\!]$ 5760	$[\![9,4,2,1,+]\!]$ 313600	$[\![9,4,2,1,-]\!]$ 313600	$[\![9,5,2,+]\!]$ 52480	$[\![9,5,2,-]\!]$ 52480
$\star$	5760	$\langle\!\langle 14,2,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	.
$\star$	5760	$\langle\!\langle 14,2,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	.
$\star$	58240	$\langle\!\langle 12,4,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	1
$\star$	58240	$\langle\!\langle 12,4,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	1	.
	732160	$\langle\!\langle 10,4,2\rangle\!\rangle$	.	.	.	.	.	.	1	1	1	1
$\star$	91520	$\langle\!\langle 9,7,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	1
$\star$	91520	$\langle\!\langle 9,7,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	1	.
	1198080	$\langle\!\langle 9,5,2\rangle\!\rangle$	1	1	1	1	1	1	1	1	1	1
$\star$	501760	$\langle\!\langle 9,4,2,1,+\rangle\!\rangle$	.	.	1	.	.	.	.	1	.	.
$\star$	501760	$\langle\!\langle 9,4,2,1,-\rangle\!\rangle$	.	.	.	1	.	.	1	.	.	.
	465920	$\langle\!\langle 7,5,4\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.
$\star$	266240	$\langle\!\langle 7,4,3,2,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.	.
$\star$	266240	$\langle\!\langle 7,4,3,2,-\rangle\!\rangle$	1	1	.	1	.	.	.	.	.	.

15 The 7-modular spin characters of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

There are four spin 7-blocks of $\tilde{S}_{16}$ and five spin 7-blocks of $\tilde{A}_{16}$ that have defect 0. The constants of the spin 7-blocks of non-zero defect are given in Table 15.1.

Table 15.1: Spin 7-blocks of non-zero defect of $\tilde{S}_{16}$ and $\tilde{A}_{16}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
68	22	18	2	(2)	2	40	17	9
69	5	3	1	$(6,3)$	1	41	7	6
70	5	3	1	$(5,4)$	1	42	7	6
71	7	6	1	$(5,3,1)$	1	43	5	3
72	5	3	1	$(8,1)$	1	44	7	6

The 7-modular basic spin characters of $\tilde{S}_{16}$ are $\sigma(16) := \sigma\langle\langle 16,+\rangle\rangle|_{7'}$, $\sigma(16)^a = \sigma\langle\langle 16,-\rangle\rangle|_{7'}$. They restrict irreducibly to $\lambda(16) := \lambda\langle\langle 16\rangle\rangle|_{7'}$ for $\tilde{A}_{16}$.

15.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{69}, B_{70}, B_{71}, and B_{72} are given in Appendix A as Figures A.19, A.20, A.21, and A.22, respectively. For the Brauer trees of the corresponding blocks b_{41}, b_{42}, b_{43}, and b_{44}, see Figures A.23, A.24, A.25, and A.26, respectively.

15.2 The block b_{40}

We have a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1,\dots,9\}$ defined by Table 15.2. We see that $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1,\dots,8\}$. Then $\lambda_{10} = -4\underline{\lambda}_2 - 3\underline{\lambda}_3 + 3\underline{\lambda}_5 + \lambda_9$ and $\lambda_{11} = -2\underline{\lambda}_1 - 4\underline{\lambda}_2 - 2\underline{\lambda}_3 + \underline{\lambda}_6 + \underline{\lambda}_8 + \lambda_9$ lead to $\underline{\lambda}_9 := \lambda_9 - 2\underline{\lambda}_1 - 4\underline{\lambda}_2 - 3\underline{\lambda}_3$ which is irreducible too. The decomposition matrix of b_{40} is Table 15.6.

Table 15.2: Some characters for b_{40}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(16)$	128	$\lambda(16) \otimes \Lambda[14, 1^2] \mid_{b_{40}}$	3342976
2	$\lambda[\![8, 6, 1, +]\!]\uparrow^{\tilde{A}_{16}}$	5632	$\lambda(16) \otimes \Lambda[8, 2, 1^6, -] \mid_{b_{40}}$	802816
3	$\lambda[\![9, 5, + : 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	16896	$\Lambda[\![9, 5, 1, +]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	2759680
4	$\lambda\langle\!\langle 6, 4, 3, 2, 1\rangle\!\rangle\mid_{7'}$	73216	$\Lambda[\![6, 4, 3, + : 2, 1]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	3286528
5	$\lambda[\![7, 5, 2, 1, -]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	85760	$\Lambda[\![7, 6, 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	3449600
6	$\lambda[\![7, 4, 3, 1, -]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	193024	$\Lambda[\![6, 3, 2, + : 3, 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}/2$	4126976
7	$\lambda[\![10, 4, + : 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	204288	$\Lambda[\![10, 4, 1, +]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	953344
8	$\lambda[\![8, 4, 3, +]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	510976	$\lambda\langle\!\langle 10, 5, 3\rangle\!\rangle\downarrow_{\tilde{A}_{16}} \mid_{b_{40}}/4$	2634240
9	$\lambda[\![7, 2, + : 5, 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	550656	$\Lambda[\![8, 5, 2, +]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}/2$	2772224
10	$\lambda[\![7, 6, 2]\!]\uparrow^{\tilde{A}_{16}} \mid_{b_{40}}$	734720		
11	$\lambda\langle\!\langle 9, 5, 2\rangle\!\rangle\mid_{7'}$	1198080		

Table 15.3: Mutual scalar products (b_{40})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9
λ_1	1	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.
λ_3	.	.	1	.	.	.	.	.	.
λ_4	.	.	.	1	.	.	.	.	.
λ_5	.	.	.	.	1	.	.	.	.
λ_6	.	.	.	.	.	1	.	.	.
λ_7	.	.	.	.	.	.	1	.	.
λ_8	.	.	.	.	.	.	.	1	.
λ_9	2	4	3	.	.	.	.	.	1

15.3 The block B_{68}

Basic sets of Brauer characters σ_i and projective characters Σ_i for $i \in \{1, \dots, 18\}$ are defined by Table 15.4. Set $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$.

Table 15.4: Some characters for B_{68}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(16)$	128	$\sigma(16) \otimes \Sigma[4, 2^6] \mid_{B_{68}}$	3342976
2	$\sigma(16)^a$	128	$\sigma(16) \otimes \Sigma[3^2, 2^5] \mid_{B_{68}}$	3342976
3	$\sigma[\![7, 6, 1 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	5888	$\Sigma[\![9, 6, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	802816
4	$\sigma[\![7, 6, 1 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	5888	$\Sigma[\![9, 5 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	3562496
5	$\sigma[\![9, 5 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	16896	$\Sigma[\![9, 4, - : 3]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	6196736
6	$\sigma[\![9, 5 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	16896	$\Sigma[\![8, 6 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	5180672
7	$\sigma\langle\!\langle 6, 4, 3, 2, 1, -\rangle\!\rangle\mid_{7'}$	73216	$\Sigma[\![9, 4, + : 3]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	6196736
8	$\sigma\langle\!\langle 6, 4, 3, 2, 1, +\rangle\!\rangle\mid_{7'}$	73216	$\Sigma[\![9, 6, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	802816
9	$\sigma[\![7, 2 : 6, 1, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	103168	$\Sigma[\![9, 5, 1]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	5519360
10	$\sigma[\![7, 2 : 6, 1, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	103168	$\Sigma[\![6, 4, 3, 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	3286528
11	$\sigma[\![10, 4 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	204288	$\Sigma[\![6, 4, 3, 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	3286528
12	$\sigma[\![10, 4 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	204288	$\Sigma[\![7, 6, 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	3449600
13	$\sigma[\![7, 4, 3 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	510976	$\Sigma[\![7, 4, 3 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	12983040
14	$\sigma[\![7, 4, 3 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	510976	$\Sigma[\![7, 6, 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	3449600
15	$\sigma[\![7, 2 : 5, 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	550656	$\Sigma[\![7, 4, 3 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	12983040
16	$\sigma[\![7, 2 : 5, 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	550656	$\Sigma[\![10, 4 : 2, -]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	953344
17	$\underline{\lambda}_6(b_{40})\uparrow^{\tilde{S}_{16}}$	386048	$\Sigma[\![10, 4 : 2, +]\!]\uparrow^{\tilde{S}_{16}} \mid_{B_{68}}$	953344
18	$\sigma\langle\!\langle 9, 4, 3, -\rangle\!\rangle\mid_{7'}$	704000	$\Lambda_9(b_{40})\uparrow^{\tilde{S}_{16}}$	5544448

We have $\underline{\lambda}_i(b_{40})\uparrow^{\tilde{S}_{16}} = \sigma_{2i-1} + \sigma_{2i}$ for $i \in \{3, 4\}$ and $\underline{\lambda}_{j+1}(b_{40})\uparrow^{\tilde{S}_{16}} = \sigma_{2j-1} + \sigma_{2j}$ for $j \in \{6, 7\}$, thus $\underline{\sigma}_k := \sigma_k$ is irreducible for $k \in \{5, \dots, 8, 11 \dots, 14\}$. As $\underline{\lambda}_6(b_{40})\uparrow^{\tilde{S}_{16}} = \sigma_{17} = \Sigma_{13}^* + \Sigma_{15}^*$, the Brauer atoms $\underline{\sigma}_{17} := \Sigma_{13}^*$ and $\underline{\sigma}_{18} := \Sigma_{15}^*$ are irreducible Brauer characters. Moreover,

$$\underline{\lambda}_2(b_{40})\uparrow^{\tilde{S}_{16}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 + \sigma_3 + \sigma_4$$

and Table 15.5 imply that $\underline{\sigma}_i := \sigma_i - \underline{\sigma}_1 - \underline{\sigma}_2$ is irreducible for $i \in \{3, 4\}$. Then $\mathrm{IBr}(B_{68})$

is completed by means of the relations

$$\underline{\lambda}_5(b_{40})\uparrow^{\tilde{S}_{16}} = -4\underline{\sigma}_1 - 4\underline{\sigma}_2 - 3\underline{\sigma}_3 - 3\underline{\sigma}_4 + \sigma_9 + \sigma_{10},$$
$$\underline{\lambda}_9(b_{40})\uparrow^{\tilde{S}_{16}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 4\underline{\sigma}_3 - 4\underline{\sigma}_4 - 3\underline{\sigma}_5 - 3\underline{\sigma}_6 + \sigma_{15} + \sigma_{16}$$

which give

$$\begin{aligned}
\underline{\sigma}_9 &:= \sigma_9 - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_3 - 2\underline{\sigma}_4,\\
\underline{\sigma}_{10} &:= \sigma_{10} - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 2\underline{\sigma}_3 - \underline{\sigma}_4,\\
\underline{\sigma}_{15} &:= \sigma_{15} - \underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_3 - 2\underline{\sigma}_4 - 2\underline{\sigma}_5 - \underline{\sigma}_6,\\
\underline{\sigma}_{16} &:= \sigma_{16} - \underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_3 - 2\underline{\sigma}_4 - \underline{\sigma}_5 - 2\underline{\sigma}_6.
\end{aligned}$$

The decomposition matrix of B_{68} is Table 15.7.

Table 15.5: Mutual scalar products (B_{68})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	1	1	1	1	1	2	.	.	.	.	.	.	.	.	.	.	.	.
σ_4	1	1	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.
σ_5	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.
σ_6	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.
σ_7	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
σ_8	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
σ_9	2	2	1	1	1	4	2	2	.	.	.	1	1	.	.	.	.	.
σ_{10}	2	2	2	2	2	5	1	1	.	.	.	.	.	1	1	.	.	.
σ_{11}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
σ_{12}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
σ_{13}	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.
σ_{14}	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.
σ_{15}	1	1	2	3	3	6	4	2	3	.	.	.	1	.	.	.	.	1
σ_{16}	1	1	2	4	4	7	3	2	3	.	.	.	.	.	1	.	.	1
σ_{17}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.
σ_{18}	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	.	.	.

Table 15.6: 7-modular decomposition matrix of block b_{40} of $\tilde{A}_{16}$ (defect 2, bar core (2), weight 2, content $(5,5,4,2)$)

			$[\![6,4,3,2,1]\!]$ 73216	$[\![7,4,3,2]\!]$ 193024	$[\![7,6,2,1]\!]$ 85760	$[\![7^2,2]\!]$ 128	$[\![8,6,2]\!]$ 477184	$[\![9,4,3]\!]$ 510976	$[\![9,5,2]\!]$ 16896	$[\![9,6,1]\!]$ 5632	$[\![10,4,2]\!]$ 204288
$\star$	128	$\langle\!\langle 16\rangle\!\rangle$	.	.	.	1	.	.	.	.	.
$\star$	5760	$\langle\!\langle 14,2,+\rangle\!\rangle$	.	.	.	1	.	.	.	1	.
	5760	$\langle\!\langle 14,2,-\rangle\!\rangle$	.	.	.	1	.	.	.	1	.
$\star$	22528	$\langle\!\langle 13,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
$\star$	221184	$\langle\!\langle 11,3,2\rangle\!\rangle$	.	.	.	.	.	.	1	.	1
	732160	$\langle\!\langle 10,4,2\rangle\!\rangle$	.	.	.	.	.	1	1	.	1
$\star$	91520	$\langle\!\langle 9,7,+\rangle\!\rangle$	.	.	1	1	.	.	.	1	.
	91520	$\langle\!\langle 9,7,-\rangle\!\rangle$	.	.	1	1	.	.	.	1	.
$\star$	585728	$\langle\!\langle 9,6,1\rangle\!\rangle$	.	.	1	2	1	.	1	1	.
	1198080	$\langle\!\langle 9,5,2\rangle\!\rangle$	.	1	.	.	1	1	1	.	.
$\star$	704000	$\langle\!\langle 9,4,3\rangle\!\rangle$	.	1	.	.	.	1	.	.	.
	988416	$\langle\!\langle 8,6,2\rangle\!\rangle$	2	1	2	2	1	.	.	.	.
	352000	$\langle\!\langle 7,6,2,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
	352000	$\langle\!\langle 7,6,2,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
$\star$	266240	$\langle\!\langle 7,4,3,2,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
	266240	$\langle\!\langle 7,4,3,2,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
$\star$	73216	$\langle\!\langle 6,4,3,2,1\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 15.7: 7-modular decomposition matrix of block B_{68} of $\tilde{S}_{16}$
(defect 2, bar core (2), weight 2, content (5, 5, 4, 2))

			⟦6, 4, 3, 2, 1, +⟧ 73216	⟦6, 4, 3, 2, 1, −⟧ 73216	⟦7, 4, 3, 2, +⟧ 193024	⟦7, 4, 3, 2, −⟧ 193024	⟦7, 6, 2, 1, +⟧ 85760	⟦7, 6, 2, 1, −⟧ 85760	⟦7^2, 2, +⟧ 128	⟦7^2, 2, −⟧ 128	⟦8, 6, 2, +⟧ 477184	⟦8, 6, 2, −⟧ 477184	⟦9, 4, 3, +⟧ 510976	⟦9, 4, 3, −⟧ 510976	⟦9, 5, 2, +⟧ 16896	⟦9, 5, 2, −⟧ 16896	⟦9, 6, 1, +⟧ 5632	⟦9, 6, 1, −⟧ 5632	⟦10, 4, 2, +⟧ 204288	⟦10, 4, 2, −⟧ 204288
⋆	128	⟪16, +⟫	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
⋆	128	⟪16, −⟫	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
⋆	11520	⟪14, 2⟫	.	.	.	.	.	.	1	1	.	.	.	.	.	.	1	1	.	.
⋆	22528	⟪13, 2, 1, +⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
⋆	22528	⟪13, 2, 1, −⟫	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
⋆	221184	⟪11, 3, 2, +⟫	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1
⋆	221184	⟪11, 3, 2, −⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.
⋆	732160	⟪10, 4, 2, +⟫	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.
	732160	⟪10, 4, 2, −⟫	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1
⋆	183040	⟪9, 7⟫	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	1	.	.
⋆	585728	⟪9, 6, 1, +⟫	.	.	.	.	.	1	1	1	1	.	.	.	.	1	1	.	.	.
⋆	585728	⟪9, 6, 1, −⟫	.	.	.	.	1	.	1	1	.	1	.	.	1	.	.	1	.	.
⋆	1198080	⟪9, 5, 2, +⟫	.	.	1	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.
	1198080	⟪9, 5, 2, −⟫	.	.	.	1	.	.	.	.	1	.	.	1	.	1	.	.	.	.
⋆	704000	⟪9, 4, 3, +⟫	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.
⋆	704000	⟪9, 4, 3, −⟫	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
⋆	988416	⟪8, 6, 2, +⟫	1	1	1	.	1	1	1	1	.	1	.	.	.	.	.	.	.	.
	988416	⟪8, 6, 2, −⟫	1	1	.	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.
	704000	⟪7, 6, 2, 1⟫	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	532480	⟪7, 4, 3, 2⟫	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	73216	⟪6, 4, 3, 2, 1, +⟫	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	73216	⟪6, 4, 3, 2, 1, −⟫	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

16 The 3-modular spin characters of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

The constants of the spin 3-blocks of $\tilde{S}_{17}$ and $\tilde{A}_{17}$ are given in Table 16.1.

Table 16.1: Spin 3-blocks of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
9	36	7	6	(2)	5	6	36	14
10	21	10	5	$(4,1)$	4	7	21	5

The 3-modular basic spin characters of $\tilde{S}_{17}$ is $\sigma(17) := \sigma\langle\langle 17\rangle\rangle|_{3'}$, and the 3-modular basic spin characters of $\tilde{A}_{17}$ are $\lambda(17) := \lambda\langle\langle 17,+\rangle\rangle|_{3'}$, $\lambda(17)^c = \lambda\langle\langle 17,-\rangle\rangle|_{3'}$. Furthermore, $\sigma(16,1) := \sigma\langle\langle 16,1,+\rangle\rangle|_{3'}$ and $\sigma(16,1)^a = \sigma\langle\langle 16,1,-\rangle\rangle|_{3'}$ are irreducible Brauer characters of $\tilde{S}_{17}$ whose restriction to $\tilde{A}_{17}$ is $\lambda(16,1) := \lambda\langle\langle 16,1\rangle\rangle|_{3'}$.

16.1 The block b_7

We begin with a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1,\dots,7\}$ defined by Table 16.2.

Table 16.2: Some characters for b_7

i	λ_i	$\deg\lambda_i$	Λ_i	$\deg\Lambda_i$
1	$\lambda(16,1)$	1920	$\Sigma[\![3^4,2,1,-]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}/3$	161896320
2	$\lambda(17)^c \otimes \lambda[9,4^2,-]\ \|_{b_7}$	161280	$\Sigma[\![5,3^3,1,-]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}/6$	26873856
3	$\lambda\langle\langle 7,4,3,2,1,-\rangle\rangle\|_{3'}$	339456	$\Lambda[\![7,4,3,2]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}$	25380864
4	$\lambda(17)\otimes\lambda[10,7]\ \|_{b_7}$	437376	$\Lambda[\![6,5,4,1]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}/4$	30233088
5	$\lambda[\![7,5,3,1]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}$	492288	$\Lambda[\![7,5,2:2,1]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}/6$	25380864
6			$\Lambda[\![3^5,1]\!]\uparrow^{\tilde{A}_{17}}\|_{b_7}$	371475072

Table 16.3 shows that $\underline{\Lambda}_i := \Lambda_i$ is indecomposable for $i \in \{2,\dots,4\}$. We decompose

$\Lambda_6 = 3\Lambda_1 - 2\underline{\Lambda}_2 - 2\underline{\Lambda}_4$ and get $\Lambda_{1a} := \Lambda_1 - \underline{\lambda}_2 - \underline{\Lambda}_4$. In particular, $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{1,3,4,5\}$. In 16.5 we will prove using condensation that λ_2 is irreducible too.

Table 16.3: Mutual scalar products (b_7)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
λ_1	1	.	.	.	.
λ_2	2	1	.	.	.
λ_3	.	.	1	.	.
λ_4	1	.	.	1	.
λ_5	.	.	.	.	1

16.2 The block B_{10}

Consider the basic set of Brauer characters σ_i and the basic set of projective characters Σ_i for $i \in \{1,\dots,10\}$ defined by Table 16.4.

Table 16.4: Some characters for B_{10}

i	σ_i	$\deg\sigma_i$	Σ_i	$\deg\Sigma_i$
1	$\sigma(16,1)$	1920	$\Lambda_{1a}(b_7)\uparrow^{\tilde{S}_{16}}$	209578752
2	$\sigma(16,1)^a$	1920	$\Sigma[\![3^5,1,-]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	371475072
3	$\sigma\langle\!\langle 13,4,-\rangle\!\rangle\mid_{3'}$	161280	$\Sigma[\![5,3^3,1:2]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}/6$	53747712
4	$\sigma\langle\!\langle 13,4,+\rangle\!\rangle\mid_{3'}$	161280	$\Sigma[\![7,4,3,2,+]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	25380864
5	$\sigma[\![6,4,3,2,1,+]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	339456	$\Sigma[\![7,4,3,2,-]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	25380864
6	$\sigma[\![6,5,3,2]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	339456	$\Sigma[\![6,5,4,1,-]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}/2$	60466176
7	$\sigma\langle\!\langle 10,7,-\rangle\!\rangle\mid_{3'}$	439296	$\Sigma[\![7,5,2,1:2]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}/3$	50761728
8	$\sigma\langle\!\langle 10,7,+\rangle\!\rangle\mid_{3'}$	439296	$\Sigma[\![7,5,3,1,+]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	76142592
9	$\sigma[\![7,5,3,1,+]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	439296	$\Sigma[\![3^4,1:3,1,-]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	2194138368
10	$\sigma[\![7,5,3,1,-]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	492288	$\Sigma[\![3^3,1,+:3^2,1]\!]\uparrow^{\tilde{S}_{17}}\mid_{B_{10}}$	3964453632

First of all, we have $\underline{\lambda}_i(b_7)\uparrow^{\tilde{S}_{17}} = \sigma_{2i-1} + \sigma_{2i}$ for $i \in \{1,3,5\}$. Thus $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1,2,5,6,9,10\}$. We also have $\lambda_2(b_7)\uparrow^{\tilde{S}_{17}} = \sigma_3 + \sigma_4$ but recall that we do not yet know which of $\lambda_2(b_7)$ or $\lambda_2(b_7) - \underline{\lambda}_1(b_7)$ is irreducible. Table 16.5 does not help to decide this.

Table 16.5: Mutual scalar products (B_{10})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}
σ_1	1	1	.	.	.	.	.	.	3	3
σ_2	1	2	.	.	.	.	.	.	3	3
σ_3	1	2	1	.	.	.	.	.	10	17
σ_4	1	2	1	.	.	.	.	.	9	17
σ_5	.	.	.	1	.	.	.	.	12	18
σ_6	.	.	.	.	1	.	.	.	12	18
σ_7	1	2	.	.	.	1	.	.	8	16
σ_8	1	2	.	.	.	1	.	.	8	15
σ_9	.	.	.	.	.	.	1	2	6	18
σ_{10}	.	.	.	.	.	.	1	1	6	18

Also, we cannot deduce the constituents of σ_7 and σ_8 from $\lambda_4(b_7)\uparrow^{\tilde{S}_{17}} = -\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_7 + \sigma_8$. Even the knowledge that $\underline{\Sigma}_5 := \Sigma_4$, $\underline{\Sigma}_6 := \Sigma_5$, and $\underline{\Sigma}_i := \sigma_i^*$ for $i \in \{3, 4, 7, 8, 9, 10\}$ are projective indecomposable characters does not improve the situation here. See 16.5.

16.3 The block B_9

We have basic sets $\{\sigma_1, \dots, \sigma_7\}$ and $\{\Sigma_1, \dots, \Sigma_7\}$ that are defined by Table 16.6.

Table 16.6: Some characters for B_9

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(17)$	256	$\Sigma[\![3^5, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	439126272
2	$\sigma[\![4, 3^3, 2, 1, -]\!]\uparrow^{\tilde{S}_{17}}$	13056	$\Sigma[\![5, 3^3, 2]\!]\uparrow^{\tilde{S}_{17}}$	368022528
3	$\sigma[\![5, 4, 3^2, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	43008	$\Sigma[\![5, 4, 3^2, 1-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	120932352
4	$\sigma[\![6, 5, 4, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	182784	$\underline{\Sigma}_9(B_{10}) \otimes \sigma[16, 1]\vert_{B_9}$	234399744
5	$\sigma[\![6, 4, 3, 2, 1, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	269824	$\Sigma[\![6, 5, 3, 2]\!]\uparrow^{\tilde{S}_{17}}$	155457792
6	$\sigma[\![3^3, 1, + : 4, 2, 1]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	1570560	$\Sigma[\![7, 4, 3, 2, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}/2$	28553472
7	$\sigma[\![7, 5, 3, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	2292992	$\Sigma[\![8, 5, 2, 1]\!]\uparrow^{\tilde{S}_{17}}/2$	69797376
8	$\sigma[\![6, 5, 3, 2]\!]\uparrow^{\tilde{S}_{17}}$	3368448	$\Sigma[\![4, 3^3, 2, 1, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_9}$	526652928
9			$\underline{\Sigma}_7(B_{10}) \otimes \sigma[8^2, 1]\vert_{B_9}/2$	136048896

By Table 16.7, $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1, \ldots, 5\}$ and $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{5, 6, 7\}$.

Table 16.7: Mutual scalar products (B_9)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
σ_1	1	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.
σ_3	.	.	1	.	.	.	.
σ_4	.	.	.	1	.	.	.
σ_5	.	.	.	.	1	.	.
σ_6	4	1	2	1	.	1	.
σ_7	.	1	.	1	.	.	1

Then $\sigma_8 = -2\underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_3 + 3\underline{\sigma}_4 + 5\underline{\sigma}_5 + \sigma_6$ yields $\sigma_{6a} := \sigma_6 - 2\underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_3$. Hence $\underline{\Sigma}_3 := \Sigma_3$ is indecomposable. Since $\Sigma_8 = \Sigma_2 + 8\underline{\Sigma}_6 - \underline{\Sigma}_7$ and $\Sigma_9 = \Sigma_4 - \underline{\Sigma}_6 - \underline{\Sigma}_7$, we get $\underline{\Sigma}_2 := \Sigma_2 - \underline{\Sigma}_7$ and $\underline{\Sigma}_4 := \Sigma_4 - \underline{\Sigma}_6 - \underline{\Sigma}_7$ which are indecomposable. In particular, $\underline{\sigma}_7 := \sigma_7$ is irreducible. We proceed with the block b_6 that is covered by B_9. We will use condensation in 16.6 to show that σ_{6a} is irreducible too.

16.4 The block b_6

Our basic sets of Brauer characters λ_i and projective characters Λ_i for $i \in \{1, \ldots, 14\}$ are defined by Table 16.8. Let $\underline{\lambda}_i := \lambda_i$ for $i \in \{1, 2\}$.
We have $\underline{\lambda}_i{}^{\circledast} = \underline{\lambda}_i$ for $i \in \{1, 2\}$. As $\underline{\sigma}_2(B_9)\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 + \lambda_3 + \lambda_4$ and $\lambda_3^{\circledast} = \lambda_4$, it follows that $\underline{\lambda}_i := \lambda_i - \underline{\lambda}_1 - \underline{\lambda}_2$ is irreducible for $i \in \{3, 4\}$. Similarly,

$$\underline{\sigma}_3(B_9)\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - \underline{\lambda}_3 - \underline{\lambda}_4 + \lambda_5 + \lambda_6$$

with $\lambda_5^{\circledast} = \lambda_6$, and the decompositions in terms of the Brauer atoms determine the irreducible Brauer characters $\underline{\lambda}_5 := \lambda_5 - \underline{\lambda}_1 - \underline{\lambda}_2 - \underline{\lambda}_4$ and $\underline{\lambda}_6 := \lambda_6 - \underline{\lambda}_1 - \underline{\lambda}_2 - \underline{\lambda}_3$. Consider

$$\begin{aligned}
\underline{\sigma}_5(B_9)\downarrow_{\tilde{A}_{17}} &= -6\underline{\lambda}_1 - 6\underline{\lambda}_2 - 5\underline{\lambda}_3 - 5\underline{\lambda}_4 - 3\underline{\lambda}_5 - 3\underline{\lambda}_6 + \lambda_8 + \lambda_9, \\
\sigma_{6a}(B_9)\downarrow_{\tilde{A}_{17}} &= -8\underline{\lambda}_1 - 8\underline{\lambda}_2 - 6\underline{\lambda}_3 - 6\underline{\lambda}_4 - 5\underline{\lambda}_5 - 5\underline{\lambda}_6 + \lambda_{11} + \lambda_{12}, \\
\underline{\sigma}_7(B_9)\downarrow_{\tilde{A}_{17}} &= -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - \underline{\lambda}_3 - \underline{\lambda}_4 - \lambda_7 + \lambda_{13} + \lambda_{14}.
\end{aligned}$$

Table 16.8: Some characters for $\mathfrak{b}_6$

i	λ_i	$\deg\lambda_i$	Λ_i	$\deg\Lambda_i$
1	$\lambda(17)$	128	$\Lambda[\![3^5,1]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	439126272
2	$\lambda(17)^c$	128	$\lambda(17)^c\otimes\Lambda[9,4^2,-]\|_{\mathfrak{b}_6}$	2319083136
3	$\sigma[\![3^4,2,1,-]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	6784	$\Lambda[\![5,3^3,2,+]\!]\uparrow^{\tilde{A}_{17}}$	184011264
4	$\sigma[\![3^4,2,1,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	6784	$\Lambda[\![5,3^3,2,-]\!]\uparrow^{\tilde{A}_{17}}$	184011264
5	$\lambda\langle\!\langle 14,2,1,+\rangle\!\rangle\|_{3'}$	28288	$\Sigma[\![5,3^3,1,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	365409792
6	$\lambda\langle\!\langle 14,2,1,-\rangle\!\rangle\|_{3'}$	28288	$\Lambda[\![5,4,3^2,1]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	120932352
7	$\lambda(17)\otimes\lambda[10,6,1]\|_{\mathfrak{b}_6}$	182784	$\Sigma[\![6,5,3,1,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	421396992
8	$\lambda[\![3^3,2,1,+:3,2,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	232832	$\Lambda[\![6,4,2,+:3,2,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	1579398912
9	$\lambda[\![3^3,2,1,+:3,2,-]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	232832	$\Lambda[\![6,5,3,2,+]\!]\uparrow^{\tilde{A}_{17}}$	77728896
10	$\lambda[\![3^3,2,+:3,2,1,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	628864	$\Sigma[\![6,4,3,2,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	304570368
11	$\lambda[\![3^3,2,+:4,2,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	883200	$\Lambda[\![6,5,3,2,-]\!]\uparrow^{\tilde{A}_{17}}$	77728896
12	$\lambda[\![3^3,2,+:4,2,-]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	883200	$\underline{\Lambda}_4(\mathfrak{b}_7)\otimes\lambda[16,1]\|_{\mathfrak{b}_6}/2$	136048896
13	$\lambda\langle\!\langle 8,7,2,+\rangle\!\rangle\|_{3'}$	1244672	$\Lambda[\![7,4,3,2]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}/2$	28553472
14	$\lambda\langle\!\langle 8,7,2,-\rangle\!\rangle\|_{3'}$	1244672	$\Lambda[\![8,5,2,1,-]\!]\uparrow^{\tilde{A}_{17}}$	69797376
15	$\sigma[\![6,5,3,1,+]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	1420928		
16	$\sigma[\![6,5,3,1,-]\!]\uparrow^{\tilde{A}_{17}}\|_{\mathfrak{b}_6}$	1420928		

It follows from ${\lambda_i}^{\circledast}=\lambda_{i+1}$ for $i\in\{8,11,13\}$ and Table 16.9 that

$$\begin{aligned}
\underline{\lambda}_8 &:= \sigma_8-3\underline{\lambda}_1-3\underline{\lambda}_2-2\underline{\lambda}_3-3\underline{\lambda}_4-2\underline{\lambda}_5-\underline{\lambda}_6,\\
\underline{\lambda}_9 &:= \sigma_9-3\underline{\lambda}_1-3\underline{\lambda}_2-3\underline{\lambda}_3-2\underline{\lambda}_4-\underline{\lambda}_5-2\underline{\lambda}_6,\\
\lambda_{11a} &:= \sigma_{11}-4\underline{\lambda}_1-4\underline{\lambda}_2-3\underline{\lambda}_3-3\underline{\lambda}_4-2\underline{\lambda}_5-3\underline{\lambda}_6,\\
\lambda_{12a} &:= \sigma_{12}-4\underline{\lambda}_1-4\underline{\lambda}_2-3\underline{\lambda}_3-3\underline{\lambda}_4-3\underline{\lambda}_5-2\underline{\lambda}_6,\\
\lambda_{13a} &:= \sigma_{13}-\underline{\lambda}_1-\underline{\lambda}_2,\\
\lambda_{14a} &:= \sigma_{14}-\underline{\lambda}_1-\underline{\lambda}_2.
\end{aligned}$$

Then $\underline{\lambda}_8$ and $\underline{\lambda}_9$ are irreducible. Moreover, the relations $\lambda_{15}=\underline{\lambda}_1+\underline{\lambda}_2-\underline{\lambda}_4+\lambda_7+\lambda_{14a}$ and $\lambda_{16}=\underline{\lambda}_1+\underline{\lambda}_2-\underline{\lambda}_3+\lambda_7+\lambda_{13a}$ give $\lambda_{13b}:=\sigma_{13a}-\underline{\lambda}_3$ and $\lambda_{14b}:=\sigma_{14a}-\underline{\lambda}_4$. The projective characters $\underline{\Lambda}_9:=\Lambda_9$ and $\underline{\Lambda}_{10}:=\Lambda_{11}$ are indecomposable since their sum equals $\underline{\Sigma}_5(B_9)\downarrow_{\tilde{A}_{17}}$. As $\underline{\Sigma}_6(B_9)\downarrow_{\tilde{A}_{17}}=\Lambda_{13}=\lambda_{11}^*+\lambda_{12}^*$ and $\underline{\Sigma}_7(B_9)\downarrow_{\tilde{A}_{17}}=\Lambda_{14}=\lambda_{13}^*+\lambda_{14}^*$, the projective atoms $\underline{\Lambda}_i:=\lambda_i^*$, $i\in\{11,\dots,14\}$, are projective indecomposable characters. Then $\underline{\Sigma}_2(B_9)\downarrow_{\tilde{A}_{17}}=\Lambda_3+\Lambda_4-\underline{\Lambda}_{13}-\underline{\Lambda}_{14}$ yields $\underline{\Lambda}_3:=\Lambda_3-\underline{\Lambda}_{14}$ and $\underline{\Lambda}_4:=\Lambda_4-\underline{\Lambda}_{13}$ which are indecomposable. We use the known pairs $(\underline{\lambda}_3,\underline{\Lambda}_3)$, $(\underline{\lambda}_4,\underline{\Lambda}_4)$, $(\underline{\lambda}_8,\underline{\Lambda}_9)$, and $(\underline{\lambda}_9,\underline{\Lambda}_{10})$

Table 16.9: Mutual scalar products ($\mathfrak{b}_6$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}
λ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_2	1	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	2	1	1	.	.	.	.	.	.	.	.	.	.	.
λ_4	2	1	.	1	1	.	.	.	.	.	.	.	.	.
λ_5	2	1	.	1	2	1	.	.	.	.	.	.	.	.
λ_6	2	1	1	.	2	1	.	.	.	.	.	.	.	.
λ_7	.	4	.	.	.	.	3	3	.	.	.	2	.	.
λ_8	6	6	2	3	7	3	1	1	1	2	.	.	.	.
λ_9	6	6	3	2	7	3	.	2	.	1	1	.	.	.
λ_{10}	8	15	3	4	10	4	3	6	2	5	1	1	.	.
λ_{11}	10	16	3	3	11	5	.	7	.	3	.	.	1	.
λ_{12}	10	16	3	3	10	5	.	7	.	2	.	.	1	.
λ_{13}	2	18	1	1	1	.	3	15	.	.	.	1	.	1
λ_{14}	2	18	1	1	1	.	4	15	.	.	.	1	.	1

to improve λ_{10} by $\lambda_{10a} := \lambda_{10} - 3\underline{\lambda}_3 - 4\underline{\lambda}_4 - 2\underline{\lambda}_8 - \underline{\lambda}_9$. Likewise, $\Lambda_{2a} := \Lambda_2 - 3\underline{\Lambda}_9 - 3\underline{\Lambda}_{10}$, $\Lambda_{5a} := \Lambda_5 - \underline{\Lambda}_4$, and $\Lambda_{7a} := \Lambda_7 - \underline{\Lambda}_9$ improve Λ_2, Λ_5, and Λ_7, respectively. The new mutual scalar products are given in Table 16.10.

As $\underline{\Sigma}_3(B_9)\downarrow_{\tilde{A}_{17}} = \Lambda_6$, we aim to split Λ_6 into its conjugate constituents. Since we have $\deg \underline{\lambda}_5^* = \deg \underline{\lambda}_6^* = 60466176$ and $\deg \lambda_{10a}^* = 0$, these constituents are $\underline{\Lambda}_5(x) := \underline{\lambda}_5^* + x\lambda_{10a}^*$ and $\underline{\Lambda}_6(x) := \underline{\lambda}_6^* + (4-x)\lambda_{10a}^*$ for a suitable integer x with $0 \leqslant x \leqslant 4$. The relations

$$\begin{aligned} \Lambda_5\uparrow^{\tilde{S}_{17}} &= 3\underline{\Sigma}_3(B_9) + \underline{\Sigma}_7(B_9), \\ \Lambda_6\uparrow^{\tilde{S}_{17}} &= 2\underline{\Sigma}_3(B_9) \end{aligned}$$

imply that $y\underline{\Lambda}_5(x) + (3-y)\underline{\Lambda}_6(x)$ is contained in Λ_5 for a suitable $y \in \{0,1,2,3\}$. In addition, either $\underline{\Lambda}_{13}$ or $\underline{\Lambda}_{14}$ is contained in Λ_{5a} because $\underline{\Lambda}_i\uparrow^{\tilde{S}_{17}} = \underline{\Sigma}_7(B_9)$ for $i \in \{13, 14\}$. Then Table 16.10 determines $y = 1$ and $x = 2$. Thus $\underline{\Lambda}_5 := \underline{\Lambda}_5(2)$ and $\underline{\Lambda}_6 := \underline{\Lambda}_6(2)$ are projective indecomposable characters. Similarly, we know from $\underline{\Sigma}_4(B_9)\downarrow_{\tilde{A}_{17}} = \Lambda_8$ and by consideration of the degrees of the projective atoms that the constituents of Λ_8 are $\underline{\Lambda}_7(x) := \lambda_7^* + x\lambda_{10a}^* + \lambda_{13b}^*$ and $\underline{\Lambda}_8(y) := \lambda_7^* + y\lambda_{10a}^* + \lambda_{14b}^*$ for suitable $x, y \in \{0, 1\}$ such

Table 16.10: Mutual scalar products (b_6)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_{2a}	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	Λ_{5a}	Λ_6	Λ_{7a}	Λ_8	$\underline{\Lambda}_9$	$\underline{\Lambda}_{10}$	$\underline{\Lambda}_{11}$	$\underline{\Lambda}_{12}$	$\underline{\Lambda}_{13}$	$\underline{\Lambda}_{14}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_5$	.	.	.	.	1	1	.	.	.	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	2	1	.	.	.	.	.	.	.	.
λ_7	.	4	.	.	.	.	3	2	.	.	.	.	.	.
$\underline{\lambda}_8$	.	.	.	.	.	.	.	.	1	.	.	.	.	.
$\underline{\lambda}_9$	.	.	.	.	.	.	.	.	.	1	.	.	.	.
λ_{10a}	8	6	.	.	6	4	1	1	.	.	.	.	.	.
λ_{11b}	2	12	.	.	.	.	.	.	.	.	1	.	.	.
λ_{12b}	2	12	.	.	.	.	.	.	.	.	.	1	.	.
λ_{13b}	.	17	.	.	1	.	3	1	.	.	.	.	1	.
λ_{14b}	.	17	.	.	.	.	4	1	.	.	.	.	.	1

that $x + y = 1$. It follows from $\underline{\Lambda}_i {\uparrow}^{\tilde{S}_{17}} = \underline{\Sigma}_6(B_9)$ for $i \in \{11, 12\}$ and

$$\Lambda_{2a} {\uparrow}^{\tilde{S}_{17}} = \Sigma_1(B_9) + 4\underline{\Sigma}_4(B_9) + 22\underline{\Sigma}_6(B_9) + 30\underline{\Sigma}_7(B_9)$$

that Λ_{2a} contains Λ_8 twice, $\underline{\Lambda}_{11}$ and $\underline{\Lambda}_{12}$ at least 10 times each, as well as $\underline{\Lambda}_{13}$ and $\underline{\Lambda}_{14}$ at least 13 times each, see Table 16.10. We deduce that Λ_7 contains Λ_8 and $\underline{\Lambda}_{14}$ from

$$\Lambda_7 {\uparrow}^{\tilde{S}_{17}} = 3\underline{\Sigma}_4(B_9) + 4\underline{\Sigma}_7(B_9).$$

Hence $\Lambda_{2b} := \Lambda_{2a} - 2\Lambda_8 - 10\underline{\Lambda}_{11} - 10\underline{\Lambda}_{12} - 13\underline{\Lambda}_{13} - 13\underline{\Lambda}_{14}$ and $\Lambda_{7b} := \Lambda_7 - \Lambda_8 - \underline{\Lambda}_{14}$ improve the basic set of projective characters in exchange with Λ_{2a} and Λ_7. This enables us to split λ_7: we have $\underline{\sigma}_4(B_9){\downarrow}_{\tilde{A}_{17}} = \lambda_7 = \Lambda^*_{7b} + 2\Lambda^*_8$ with $\deg \Lambda^*_{7b} = 0$. Thus $\underline{\lambda}_7 := \Lambda^*_8$ and $\underline{\lambda}_8 := \Lambda^*_{7b} + \Lambda^*_8$ are irreducible Brauer characters. Inserting these instead of λ_7 and λ_{10a} into the basic set of Brauer characters leads to $\Lambda_{2c} := \Lambda_{2b} - 2\underline{\Lambda}_{13} - 2\underline{\Lambda}_{14}$ and $\Lambda_{7c} := \Lambda_{7b} - \underline{\Lambda}_{13} - \underline{\Lambda}_{14}$ by means of the above relations. The resulting mutual scalar products are given in Table 16.11. We are left with the questions whether or not λ_{11b} and λ_{12b} (and $\sigma_{6a} \in B_9$) are irreducible, and we have to decide which of $\{\lambda_{13b} - \underline{\lambda}_7, \lambda_{14b} - \underline{\lambda}_8\}$ or $\{\lambda_{13b} - \underline{\lambda}_8, \lambda_{13b} - \underline{\lambda}_7\}$ is a subset of $\mathrm{IBr}(b_6)$.

Table 16.11: Mutual scalar products (b_6)

$\langle \cdot,\cdot \rangle$	Λ_1	Λ_{2c}	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	$\underline{\Lambda}_5$	$\underline{\Lambda}_6$	Λ_8	Λ_{7c}	$\underline{\Lambda}_9$	$\underline{\Lambda}_{10}$	$\underline{\Lambda}_{11}$	$\underline{\Lambda}_{12}$	$\underline{\Lambda}_{13}$	$\underline{\Lambda}_{14}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_5$	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$\underline{\lambda}_7$	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$\underline{\lambda}_8$	.	.	.	.	.	.	1	1	.	.	.	.	.	.
$\underline{\lambda}_9$	.	.	.	.	.	.	.	.	1	.	.	.	.	.
$\underline{\lambda}_{10}$	.	.	.	.	.	.	.	.	.	1	.	.	.	.
λ_{11b}	2	2	.	.	.	.	.	.	.	.	1	.	.	.
λ_{12b}	2	2	.	.	.	.	.	.	.	.	.	1	.	.
λ_{13b}	.	.	.	.	.	.	1	1	.	.	.	.	1	.
λ_{14b}	.	.	.	.	.	.	1	1	.	.	.	.	.	1

16.5 Condensation for B_{10} and b_7

Let $K := \langle t_{(7,1^{10})}, t_{(1^7,7,1^3)}, \tilde{s} \rangle$ where $s := (1,8)(7,14)(6,13)(5,12)(4,11)(3,10)(2,9)$, and set $e := e_K$. Then $K \cong 7^2 : 2$. Let $256a^-$ be a basic spin module over $F := GF(9)$. We construct an $F\tilde{S}_{17}$-module $1597a^+$ that affords $\sigma[13,4]$ by chopping the sequence $(16a^+ \otimes 16a^+, 16a^+ \otimes 103a^+, 16a^+ \otimes 544a^+)$ where $16a^+$ denotes the reduced permutation module of S_{17}. The Brauer character of $V := 256a^- \otimes 1597a^+$ decomposes as

$$\sigma(17) \otimes \sigma[13,4] = \underline{\sigma}_1(B_9) + 2\underline{\sigma}_3(B_9) + \sigma_3(B_{10}) + \sigma_4(B_{10}).$$

We have $\langle \underline{\sigma}_1(B_9)\downarrow_K, 1_K \rangle = 8$, $\langle \underline{\sigma}_3(B_9)\downarrow_K, 1_K \rangle = 480$, and $\langle \sigma_i(B_{10})\downarrow_K, 1_K \rangle = 1728$ for $i \in \{3,4\}$. Let $H := eF\tilde{S}_{17}e$. Since $\langle \underline{\sigma}_i(B_{10})\downarrow_K, 1_K \rangle = 36$ for $i \in \{1,2\}$, Ve has two distinct H-composition factors that correspond to irreducible constituents of $\sigma_3(B_{10})$ and $\sigma_4(B_{10})$ such that both have either dimension 1728, or dimension 1692. Let $a := t_1$, $b := t_{16} \cdots t_1$, $c := (ab)^3 b$ and $d := (ab)^2 bab^2 c$. Our condensation algebra is $C := \langle ece, ede \rangle$. Then Ve has two distinct C-composition factors of dimension 1728. Thus $\underline{\sigma}_i(B_{10}) := \sigma_i(B_{10})$ is irreducible for $i \in \{3,4\}$. In particular, $\underline{\lambda}_2(b_7) := \lambda_2(b_7)$ is irreducible.
Furthermore, we condense $W := 256a^- \otimes 4845a^+$ and use the trace formula to identify

the missing pair of irreducibles of B_{10} (either $\{\sigma_7(B_{10}) - \underline{\sigma}_1(B_{10}), \sigma_8(B_{10}) - \underline{\sigma}_2(B_{10})\}$ or $\{\sigma_7(B_{10}) - \underline{\sigma}_2(B_{10}), \sigma_8(B_{10}) - \underline{\sigma}_1(B_{10})\}$). We construct the $F\tilde{S}_{17}$-module $4845a^+$ that affords $\sigma[10,7]$ as follows: we take the reduced permutation module $10a^+$ of S_{12} and chop the sequence $(10a^+ \otimes 10a^+, 10a^+ \otimes 45a^+)$ which produces a module $143a^+$ for S_{12}. Chopping the next sequence $(143a^+ \uparrow^{S_{13}}, 417a^+ \uparrow^{S_{14}}, 428a^+ \uparrow^{S_{15}}, 1428a^+ \uparrow^{S_{16}}, 1428b^+ \uparrow^{S_{17}})$ yields $4845a^+$. Set $x := (1,7,12,17,4,9,14)(2,6,11,16,5,10,15,3,8,13)$ and $D := \langle C, e\tilde{x}e\rangle$. Then the traces ζ_9 and ζ_9^3 of $e\tilde{x}e$ on the two distinct 4356-dimensional D-composition factors of We determine the irreducible Brauer characters $\underline{\sigma}_7(B_{10}) := \sigma_7(B_{10}) - \underline{\sigma}_2(B_{10})$ and $\underline{\sigma}_8(B_{10}) := \sigma_8(B_{10}) - \underline{\sigma}_1(B_{10})$. The decomposition matrices of B_{10} and b_7 are Tables 16.13 and 16.12, respectively.

16.6 Condensation for B_9 and b_6

Let $\underline{\sigma}_6$ denote the missing element of $\mathrm{IBr}(B_9)$. Then $\underline{\sigma}_6$ is a constituent of $\sigma_6 = \sigma_6(B_9)$ which could contain $\underline{\sigma}_1(B_9) = \sigma(17)$ at most twice, thus $\deg \underline{\sigma}_6 = 1470976 - x256$ for some $x \in \{0,1,2\}$. In the setting of 16.5, we condense the $F\tilde{S}_{17}$-module $V := 1920a^- \otimes 2108a^+$ where $1920a^-$ is a constituent of $256a^- \otimes 16a^+$ that affords $\underline{\sigma}_1(B_{10})$ or $\underline{\sigma}_2(B_{10})$, and $2108a^+$ is a constituent of $16a^+ \otimes 1597a^+$ that affords $\sigma[12,5]$. Its Brauer character is

$$\underline{\sigma}_i(B_{10}) \otimes \sigma[12,5] = 3\underline{\sigma}_4(B_9) + 5\underline{\sigma}_5(B_9) + \sigma_6 + \underline{\sigma}_5(B_{10}) + \underline{\sigma}_6(B_{10}) \quad (i \in \{1,2\}).$$

We have $\langle \underline{\sigma}_4(B_9)\downarrow_K, 1_K\rangle = 1824$, $\langle \underline{\sigma}_5(B_9)\downarrow_K, 1_K\rangle = 2660$, $\langle \sigma_6\downarrow_K, 1_K\rangle = 14936$, and $\langle \underline{\sigma}_i(B_{10})\downarrow_K, 1_K\rangle = 3420$ for $i \in \{5,6\}$. Recall that $\langle \underline{\sigma}_1(B_9)\downarrow_K, 1_K\rangle = 8$. As Ve has a C-composition factor of dimension 14936, we can deduce that $\underline{\sigma}_6 = \sigma_6$. In particular, $\underline{\lambda}_{11} := \lambda_{11b}$ and $\underline{\lambda}_{12} := \lambda_{12b}$ are irreducible Brauer characters of b_6.

We complete $\mathrm{IBr}(b_6)$ by condensing $W := 6528a^- \otimes (2108a^+\downarrow_{\tilde{A}_{17}})$ with respect to the trivial idempotent e of $K := \langle t_{(7,1^{10})}, zt_{(1^7,5,1^5)}, zt_{(1^{12},5)}, \tilde{s}_1, \tilde{s}_2\rangle \cong (7 \times 5^2) : 2^2$ where $s_1 := (9,12)(10,11)(14,17)(15,16)$ and $s_2 := (2,7)(3,6)(4,5)(8,13)(9,14)(10,15)(11,16)(12,17)$. We construct the $F\tilde{A}_{17}$-module $6528a^-$ as a constituent of $128a^- \otimes (103a^+\downarrow_{\tilde{A}_{17}})$ where $128a^-$ is a basic spin module of $\tilde{A}_{17}$. We set $a := t_2t_1$, $b := t_{16}\cdots t_1$, $c := (ab)^3b$, and $d := (ab)^2bab^2c$. Also, set $y := (1,9)(2,10,16,6,13,3,8,15,5,12)(4,11,17,7,14)$. With respect to the condensation algebra $C := \langle ece, ede, e\tilde{y}e\rangle$, the trace of $e\tilde{y}e$ on the unique 1700-dimensional C-composition factor of We that we are interested in is ζ_9. This is sufficient to identify $\lambda_{13b}(b_6) - \underline{\lambda}_8(b_6)$ and $\lambda_{14b}(b_6) - \underline{\lambda}_7(b_6)$ as elements of $\mathrm{IBr}(b_6)$. The decomposition matrices of B_9 and b_6 are Tables 16.14 and 16.15, respectively.

Table 16.12: 3-modular decomposition matrix of block b_7 of $\tilde{A}_{17}$ (defect 5, bar core $(4,1)$, weight 4, content $(12,5)$)

			$[\![4,3^4,1]\!]$ 1920	$[\![6,4,3^2,1]\!]$ 161280	$[\![6^2,4,1]\!]$ 437376	$[\![7,4,3,2,1]\!]$ 339456	$[\![7,5,4,1]\!]$ 492288
$\star$	1920	$\langle\!\langle 16,1\rangle\!\rangle$	1	.	.	.	.
$\star$	161280	$\langle\!\langle 13,4\rangle\!\rangle$	.	1	.	.	.
	163200	$\langle\!\langle 13,3,1,+\rangle\!\rangle$	1	1	.	.	.
	163200	$\langle\!\langle 13,3,1,-\rangle\!\rangle$	1	1	.	.	.
	502656	$\langle\!\langle 12,4,1,+\rangle\!\rangle$	1	1	.	1	.
	502656	$\langle\!\langle 12,4,1,-\rangle\!\rangle$	1	1	.	1	.
$\star$	439296	$\langle\!\langle 10,7\rangle\!\rangle$	1	.	1	.	.
	1272960	$\langle\!\langle 10,6,1,+\rangle\!\rangle$	2	.	1	1	1
	1272960	$\langle\!\langle 10,6,1,-\rangle\!\rangle$	2	.	1	1	1
	1436160	$\langle\!\langle 10,4,3,+\rangle\!\rangle$	3	1	1	1	1
	1436160	$\langle\!\langle 10,4,3,-\rangle\!\rangle$	3	1	1	1	1
	2036736	$\langle\!\langle 10,4,2,1\rangle\!\rangle$	4	2	2	1	1
$\star$	933504	$\langle\!\langle 9,7,1,+\rangle\!\rangle$	2	.	1	.	1
	933504	$\langle\!\langle 9,7,1,-\rangle\!\rangle$	2	.	1	.	1
	2872320	$\langle\!\langle 9,4,3,1\rangle\!\rangle$	6	2	2	2	2
	1096704	$\langle\!\langle 7,6,4,+\rangle\!\rangle$	3	1	1	.	1
	1096704	$\langle\!\langle 7,6,4,-\rangle\!\rangle$	3	1	1	.	1
	2872320	$\langle\!\langle 7,6,3,1\rangle\!\rangle$	6	2	2	2	2
	2376192	$\langle\!\langle 7,5,4,1\rangle\!\rangle$	4	2	2	2	1
$\star$	339456	$\langle\!\langle 7,4,3,2,1,+\rangle\!\rangle$	.	.	.	1	.
	339456	$\langle\!\langle 7,4,3,2,1,-\rangle\!\rangle$	.	.	.	1	.

Table 16.13: 3-modular decomposition matrix of block B_{10} of $\tilde{S}_{17}$
(defect 5, bar core $(4,1)$, weight 4, content $(12,5)$)

			$[\![4,3^4,1,+]\!]$ 1920	$[\![4,3^4,1,-]\!]$ 1920	$[\![6,4,3^2,1,+]\!]$ 161280	$[\![6,4,3^2,1,-]\!]$ 161280	$[\![6^2,4,1,+]\!]$ 437376	$[\![6^2,4,1,-]\!]$ 437376	$[\![7,4,3,2,1,+]\!]$ 339456	$[\![7,4,3,2,1,-]\!]$ 339456	$[\![7,5,4,1,+]\!]$ 492288	$[\![7,5,4,1,-]\!]$ 492288
$\star$	1920	$\langle\!\langle 16,1,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.
$\star$	1920	$\langle\!\langle 16,1,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
$\star$	161280	$\langle\!\langle 13,4,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
$\star$	161280	$\langle\!\langle 13,4,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.
	326400	$\langle\!\langle 13,3,1\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.
	1005312	$\langle\!\langle 12,4,1\rangle\!\rangle$	1	1	1	1	.	.	1	1	.	.
$\star$	439296	$\langle\!\langle 10,7,+\rangle\!\rangle$	1	.	.	.	.	1	.	.	.	.
$\star$	439296	$\langle\!\langle 10,7,-\rangle\!\rangle$	.	1	.	.	1	.	.	.	.	.
	2545920	$\langle\!\langle 10,6,1\rangle\!\rangle$	2	2	.	.	1	1	1	1	1	1
	2872320	$\langle\!\langle 10,4,3\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	1
$\star$	2036736	$\langle\!\langle 10,4,2,1,+\rangle\!\rangle$	2	2	1	1	1	1	1	.	.	1
	2036736	$\langle\!\langle 10,4,2,1,-\rangle\!\rangle$	2	2	1	1	1	1	.	1	1	.
$\star$	1867008	$\langle\!\langle 9,7,1\rangle\!\rangle$	2	2	.	.	1	1	.	.	1	1
	2872320	$\langle\!\langle 9,4,3,1,+\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	1
	2872320	$\langle\!\langle 9,4,3,1,-\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	1
	2193408	$\langle\!\langle 7,6,4\rangle\!\rangle$	3	3	1	1	1	1	.	.	1	1
	2872320	$\langle\!\langle 7,6,3,1,+\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	1
	2872320	$\langle\!\langle 7,6,3,1,-\rangle\!\rangle$	3	3	1	1	1	1	1	1	1	1
$\star$	2376192	$\langle\!\langle 7,5,4,1,+\rangle\!\rangle$	2	2	1	1	1	1	1	1	.	1
	2376192	$\langle\!\langle 7,5,4,1,-\rangle\!\rangle$	2	2	1	1	1	1	1	1	1	.
$\star$	678912	$\langle\!\langle 7,4,3,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.

Table 16.14: 3-modular decomposition matrix of block B_9 of $\tilde{S}_{17}$ (defect 6, bar core (2), weight 5, content $(11, 6)$)

			$[\![3^5, 2]\!]$ 256	$[\![5, 3^3, 2, 1]\!]$ 13056	$[\![5, 4, 3^2, 2]\!]$ 43008	$[\![6, 5, 3, 2, 1]\!]$ 269824	$[\![6, 5, 4, 2]\!]$ 182784	$[\![7, 5, 3, 2]\!]$ 1470976	$[\![8, 5, 3, 1]\!]$ 2292992
⋆	256	$\langle\!\langle 17 \rangle\!\rangle$	1	.	.	.	.	.	.
⋆	13312	$\langle\!\langle 15, 2, + \rangle\!\rangle$	1	1	.	.	.	.	.
	13312	$\langle\!\langle 15, 2, - \rangle\!\rangle$	1	1	.	.	.	.	.
⋆	56320	$\langle\!\langle 14, 3, + \rangle\!\rangle$	1	1	1	.	.	.	.
	56320	$\langle\!\langle 14, 3, - \rangle\!\rangle$	1	1	1	.	.	.	.
	56576	$\langle\!\langle 14, 2, 1 \rangle\!\rangle$	2	1	1	.	.	.	.
⋆	326144	$\langle\!\langle 12, 5, + \rangle\!\rangle$	1	1	1	1	.	.	.
	326144	$\langle\!\langle 12, 5, - \rangle\!\rangle$	1	1	1	1	.	.	.
	678912	$\langle\!\langle 12, 3, 2 \rangle\!\rangle$	4	4	2	2	.	.	.
	465920	$\langle\!\langle 11, 6, + \rangle\!\rangle$	1	1	.	1	1	.	.
	465920	$\langle\!\langle 11, 6, - \rangle\!\rangle$	1	1	.	1	1	.	.
	1980160	$\langle\!\langle 11, 5, 1 \rangle\!\rangle$	2	1	1	1	1	1	.
	2558976	$\langle\!\langle 11, 4, 2 \rangle\!\rangle$	6	4	3	2	2	1	.
	522240	$\langle\!\langle 11, 3, 2, 1, + \rangle\!\rangle$	2	2	1	1	1	.	.
	522240	$\langle\!\langle 11, 3, 2, 1, - \rangle\!\rangle$	2	2	1	1	1	.	.
	4978688	$\langle\!\langle 10, 5, 2 \rangle\!\rangle$	6	3	2	2	3	1	1
⋆	183040	$\langle\!\langle 9, 8, + \rangle\!\rangle$	1	.	.	.	1	.	.
	183040	$\langle\!\langle 9, 8, - \rangle\!\rangle$	1	.	.	.	1	.	.
	5544448	$\langle\!\langle 9, 6, 2 \rangle\!\rangle$	4	4	.	2	2	.	2
	6223360	$\langle\!\langle 9, 5, 3 \rangle\!\rangle$	8	8	2	4	2	.	2
	3620864	$\langle\!\langle 9, 5, 2, 1, + \rangle\!\rangle$	6	5	2	3	2	.	1
	3620864	$\langle\!\langle 9, 5, 2, 1, - \rangle\!\rangle$	6	5	2	3	2	.	1
⋆	2489344	$\langle\!\langle 8, 7, 2 \rangle\!\rangle$	2	1	.	.	1	.	1
	5657600	$\langle\!\langle 8, 6, 3 \rangle\!\rangle$	8	6	2	2	2	.	2
	3111680	$\langle\!\langle 8, 6, 2, 1, + \rangle\!\rangle$	4	4	1	2	1	.	1
	3111680	$\langle\!\langle 8, 6, 2, 1, - \rangle\!\rangle$	4	4	1	2	1	.	1
	3351040	$\langle\!\langle 8, 5, 4 \rangle\!\rangle$	6	5	2	2	2	.	1
	5331200	$\langle\!\langle 8, 5, 3, 1, + \rangle\!\rangle$	8	6	3	3	3	1	1
	5331200	$\langle\!\langle 8, 5, 3, 1, - \rangle\!\rangle$	8	6	3	3	3	1	1
⋆	1697280	$\langle\!\langle 8, 4, 3, 2, + \rangle\!\rangle$	2	.	1	.	1	1	.
	1697280	$\langle\!\langle 8, 4, 3, 2, - \rangle\!\rangle$	2	.	1	.	1	1	.
	2489344	$\langle\!\langle 7, 5, 3, 2, + \rangle\!\rangle$	4	2	2	2	2	1	.
	2489344	$\langle\!\langle 7, 5, 3, 2, - \rangle\!\rangle$	4	2	2	2	2	1	.
	792064	$\langle\!\langle 6, 5, 4, 2, + \rangle\!\rangle$	2	2	1	2	1	.	.
	792064	$\langle\!\langle 6, 5, 4, 2, - \rangle\!\rangle$	2	2	1	2	1	.	.
	565760	$\langle\!\langle 6, 5, 3, 2, 1 \rangle\!\rangle$	.	2	.	2	.	.	.

Table 16.15: 3-modular decomposition matrix of block b_6 of $\tilde{A}_{17}$ (defect 6, bar core (2), weight 5, content $(11, 6)$)

			$[\![3^5, 2, +]\!]$ 128	$[\![3^5, 2, -]\!]$ 128	$[\![5, 3^3, 2, 1, +]\!]$ 6528	$[\![5, 3^3, 2, 1, -]\!]$ 6528	$[\![5, 4, 3^2, 2, +]\!]$ 21504	$[\![5, 4, 3^2, 2, -]\!]$ 21504	$[\![6, 5, 3, 2, 1, +]\!]$ 134912	$[\![6, 5, 3, 2, 1, -]\!]$ 134912	$[\![6, 5, 4, 2, +]\!]$ 91392	$[\![6, 5, 4, 2, -]\!]$ 91392	$[\![7, 5, 3, 2, +]\!]$ 735488	$[\![7, 5, 3, 2, -]\!]$ 735488	$[\![8, 5, 3, 1, +]\!]$ 1146496	$[\![8, 5, 3, 1, -]\!]$ 1146496
⋆	128	$\langle\!\langle 17, +\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	128	$\langle\!\langle 17, -\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	13312	$\langle\!\langle 15, 2\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.
	56320	$\langle\!\langle 14, 3\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.
⋆	28288	$\langle\!\langle 14, 2, 1, +\rangle\!\rangle$	1	1	1	.	.	1	.	.	.	.	.	.	.	.
⋆	28288	$\langle\!\langle 14, 2, 1, -\rangle\!\rangle$	1	1	.	1	1	.	.	.	.	.	.	.	.	.
	326144	$\langle\!\langle 12, 5\rangle\!\rangle$	1	1	1	1	1	1	1	1	.	.	.	.	.	.
	339456	$\langle\!\langle 12, 3, 2, +\rangle\!\rangle$	2	2	2	2	1	1	1	1	.	.	.	.	.	.
	339456	$\langle\!\langle 12, 3, 2, -\rangle\!\rangle$	2	2	2	2	1	1	1	1	.	.	.	.	.	.
	465920	$\langle\!\langle 11, 6\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	1	.	.	.	.
⋆	990080	$\langle\!\langle 11, 5, 1, +\rangle\!\rangle$	1	1	.	1	1	.	1	.	.	1	.	1	.	.
⋆	990080	$\langle\!\langle 11, 5, 1, -\rangle\!\rangle$	1	1	1	.	.	1	.	1	1	.	1	.	.	.
⋆	1279488	$\langle\!\langle 11, 4, 2, +\rangle\!\rangle$	3	3	2	2	1	2	1	1	1	1	1	.	.	.
	1279488	$\langle\!\langle 11, 4, 2, -\rangle\!\rangle$	3	3	2	2	2	1	1	1	1	1	.	1	.	.
	522240	$\langle\!\langle 11, 3, 2, 1\rangle\!\rangle$	2	2	2	2	1	1	1	1	1	1	.	.	.	.
⋆	2489344	$\langle\!\langle 10, 5, 2, +\rangle\!\rangle$	3	3	1	2	1	1	1	1	2	1	1	.	.	1
	2489344	$\langle\!\langle 10, 5, 2, -\rangle\!\rangle$	3	3	2	1	1	1	1	1	1	2	.	1	1	.
⋆	183040	$\langle\!\langle 9, 8\rangle\!\rangle$	1	1	.	.	.	.	.	.	1	1	.	.	.	.
	2772224	$\langle\!\langle 9, 6, 2, +\rangle\!\rangle$	2	2	2	2	.	.	1	1	1	1	.	.	1	1
	2772224	$\langle\!\langle 9, 6, 2, -\rangle\!\rangle$	2	2	2	2	.	.	1	1	1	1	.	.	1	1
	3111680	$\langle\!\langle 9, 5, 3, +\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
	3111680	$\langle\!\langle 9, 5, 3, -\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
	3620864	$\langle\!\langle 9, 5, 2, 1\rangle\!\rangle$	6	6	5	5	2	2	3	3	2	2	.	.	1	1
⋆	1244672	$\langle\!\langle 8, 7, 2, +\rangle\!\rangle$	1	1	.	1	.	.	.	.	1	.	.	.	.	1
⋆	1244672	$\langle\!\langle 8, 7, 2, -\rangle\!\rangle$	1	1	1	.	.	.	.	.	.	1	.	.	1	.
	2828800	$\langle\!\langle 8, 6, 3, +\rangle\!\rangle$	4	4	3	3	1	1	1	1	1	1	.	.	1	1
	2828800	$\langle\!\langle 8, 6, 3, -\rangle\!\rangle$	4	4	3	3	1	1	1	1	1	1	.	.	1	1
	3111680	$\langle\!\langle 8, 6, 2, 1\rangle\!\rangle$	4	4	4	4	1	1	2	2	1	1	.	.	1	1
⋆	1675520	$\langle\!\langle 8, 5, 4, +\rangle\!\rangle$	3	3	2	3	1	1	1	1	1	1	.	.	.	1
	1675520	$\langle\!\langle 8, 5, 4, -\rangle\!\rangle$	3	3	3	2	1	1	1	1	1	1	.	.	1	.
	5331200	$\langle\!\langle 8, 5, 3, 1\rangle\!\rangle$	8	8	6	6	3	3	3	3	3	3	1	1	1	1
	1697280	$\langle\!\langle 8, 4, 3, 2\rangle\!\rangle$	2	2	.	.	1	1	.	.	1	1	1	1	.	.
	2489344	$\langle\!\langle 7, 5, 3, 2\rangle\!\rangle$	4	4	2	2	2	2	2	2	2	2	1	1	.	.
	792064	$\langle\!\langle 6, 5, 4, 2\rangle\!\rangle$	2	2	2	2	1	1	2	2	1	1	.	.	.	.
⋆	282880	$\langle\!\langle 6, 5, 3, 2, 1, +\rangle\!\rangle$	.	.	1	1	.	.	1	1	.	.	.	.	.	.
	282880	$\langle\!\langle 6, 5, 3, 2, 1, -\rangle\!\rangle$	.	.	1	1	.	.	1	1	.	.	.	.	.	.

17 The 5-modular spin characters of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

The block constants of the spin 5-blocks of $\tilde{S}_{17}$ and $\tilde{A}_{17}$ are given in Table 17.1.

Table 17.1: Spin 5-blocks of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
27	13	10	2	$(6,1)$	2	17	11	5
28	27	10	3	(2)	3	18	30	20
29	13	10	2	$(4,3)$	2	19	11	5
30	4	2	1	$(8,3,1)$	1	20	5	4

The 5-modular basic spin character of $\tilde{S}_{17}$ is $\sigma(17) := \sigma\langle\langle 17\rangle\rangle|_{5'}$, the 5-modular basic spin characters of $\tilde{A}_{17}$ are $\lambda(17) := \lambda\langle\langle 17,+\rangle\rangle|_{5'}$ and $\lambda(17)^c = \lambda\langle\langle 17,-\rangle\rangle|_{5'}$. Moreover, $\sigma(16,1) := \sigma\langle\langle 16,1,+\rangle\rangle|_{5'}$ and $\sigma(16,1)^a = \sigma\langle\langle 16,1,-\rangle\rangle|_{5'}$ are irreducible Brauer characters of $\tilde{S}_{17}$, and $\lambda(16,1) := \lambda\langle\langle 16,1\rangle\rangle|_{5'}$ is an irreducible Brauer character of $\tilde{A}_{17}$.

17.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{30} and b_{20} are given in Appendix A as Figures A.27 and A.28, respectively.

17.2 The block b_{17}

We find basic sets $\{\underline{\lambda}_1, \dots, \underline{\lambda}_5\}$ and $\{\Lambda_1, \dots, \Lambda_5\}$, see Table 17.2. The mutual scalar products in Table 17.3 show that $\underline{\lambda}_i$ is irreducible for $i \in \{1,2,4,5\}$ and $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{2,3,4,5\}$. The relation $\Lambda_6 = \Lambda_1 - \underline{\Lambda}_3 + 2\underline{\Lambda}_5$ leads to the projective indecomposable character $\underline{\Lambda}_1 := \Lambda_1 - \underline{\Lambda}_3$ and implies the irreducibility of $\underline{\lambda}_3$. The decomposition matrix of b_{17} is Table 17.16.

Table 17.2: Some characters for b_{17}

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(16,1)$	1920	$\Lambda\llbracket 6,5,3,+:3,+\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}$	21488000
2	$\sigma\llbracket 5,4,3,2,1,-\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}$	282880	$\Lambda\llbracket 6,4,3,1,+:2,1\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}/2$	11968000
3	$\lambda(17)\otimes\lambda[11,6]\vert_{b_{17}}$	465920	$\Lambda\llbracket 9,6,1\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}/4$	4992000
4	$\lambda\llbracket 4,3,2,1,+:5,2,-\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}$	522240	$\Lambda\llbracket 8,5,2,1\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}/3$	8160000
5	$\lambda\llbracket 6,4,3,+:3,1,+\rrbracket\uparrow^{\tilde{A}_{17}}\vert_{b_{17}}$	2302720	$\lambda(17)\otimes\Lambda[14,3]\vert_{b_{17}}/4$	5984000
6			$\lambda(17)^c\otimes\Lambda[16,1]\vert_{b_{17}}/2$	28464000

Table 17.3: Mutual scalar products (b_{17})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
$\underline{\lambda}_1$	1	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.
$\underline{\lambda}_3$	1	.	1	.	.
$\underline{\lambda}_4$	.	.	.	1	.
$\underline{\lambda}_5$	.	.	.	.	1

Table 17.4: Some characters for B_{27}

i	$\underline{\sigma}_i$	$\deg \underline{\sigma}_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(16,1)$	1920	$\underline{\Lambda}_1(b_{17})\uparrow^{\tilde{S}_{17}}$	32992000
2	$\sigma(16,1)^a$	1920	$\Sigma\llbracket 5^3,1,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	69456000
3	$\sigma\llbracket 5^2,3,2,1,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	282880	$\Sigma\llbracket 6,5,3,2,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	11968000
4	$\sigma\llbracket 5^2,3,2,1,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	282880	$\Sigma\llbracket 6,5,3,2,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	11968000
5	$\sigma\langle\langle 11,6,-\rangle\rangle\vert_{5'}$	465920	$\sigma\langle\langle 11,6,1,-\rangle\rangle\downarrow_{\tilde{S}_{17}}$	4992000
6	$\sigma\langle\langle 11,6,+\rangle\rangle\vert_{5'}$	465920	$\sigma\langle\langle 11,6,1,+\rangle\rangle\downarrow_{\tilde{S}_{17}}$	4992000
7	$\sigma\llbracket 8,5,2,1,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	522240	$\Sigma\llbracket 8,6,2,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	8160000
8	$\sigma\llbracket 8,5,2,1,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	522240	$\Sigma\llbracket 8,6,2,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	8160000
9	$\sigma\llbracket 7,5,3,1,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	2302720	$\Sigma\llbracket 7,6,3,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	5984000
10	$\sigma\llbracket 7,5,3,1,+\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	2302720	$\Sigma\llbracket 7,6,3,-\rrbracket\uparrow^{\tilde{S}_{17}}\vert_{B_{27}}$	5984000

17.3 The block B_{27}

Basic sets of Brauer characters $\underline{\sigma}_i$ and projective characters Σ_i for $i \in \{1, \ldots, 10\}$ are defined by Table 17.4. We have $\underline{\lambda}_i(b_{17}) \uparrow^{\tilde{S}_{17}} = \sigma_{2i-1} + \sigma_{2i}$ for all $i \in \{1, \ldots, 5\}$. Thus $\mathrm{IBr}(B_{27}) = \{\underline{\sigma}_1, \ldots \underline{\sigma}_{10}\}$. The decomposition matrix of B_{27} is Table 17.17.

17.4 The block b_{19}

The Brauer characters $\underline{\lambda}_i$ and the projective characters $\underline{\Lambda}_i$ defined by Table 17.5 satisfy $\langle \underline{\lambda}_i, \underline{\Lambda}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, 5\}$. The decomposition matrix of b_{19} is Table 17.18.

Table 17.5: Some characters for b_{19}

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg \underline{\Lambda}_i$
1	$\lambda(17)^c \otimes \lambda[13, 2^2] \mid_{b_{19}}$	56320	$\Lambda[\![8, 5, 3, +]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}}$	9792000
2	$\lambda[\![5^2, 3, 1, + : 3, -]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}}$	78080	$\Lambda[\![5^2, 3, 1, + : 2, 1]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}} /2$	13152000
3	$\lambda(17)^c \otimes \lambda[10, 5, 1^2] \mid_{b_{19}}$	104960	$\Sigma[\![9, 5, 1, -]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}}$	9440000
4	$\lambda[\![8, 3, 2, 1, + : 3, +]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}}$	1331200	$\lambda(17) \otimes \Lambda[9, 5, 2, 1] \mid_{b_{19}} /4$	11968000
5	$\sigma[\![6, 4, 3, 2, +]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}}$	1541120	$\Lambda[\![6, 4, 3, 1, + : 2, 1]\!] \uparrow^{\tilde{A}_{17}} \mid_{b_{19}} /2$	14144000

17.5 The block B_{29}

We begin with a basic set of Brauer characters σ_i and a basic set of projective characters Σ_i for $i \in \{1, \ldots, 10\}$ defined by Table 17.6. Then $\underline{\lambda}_1(b_{19}) \uparrow^{\tilde{S}_{17}} = \sigma_1 + \sigma_2$ shows that $\underline{\sigma}_1 := \sigma_1$ and $\underline{\sigma}_2 := \sigma_2$ are irreducible. By definition of σ_6, we know that it is the sum of two irreducible Brauer characters, say $\underline{\sigma}_5$ and $\underline{\sigma}_6$, that are associate. Since $\sigma_6 = \Sigma_2^* + \Sigma_3^* + \Sigma_4^* + 2\Sigma_6^*$ and $(\Sigma_i^*)^{\circledast} = \Sigma_i^*$ for all $i \in \{2, 3, 4, 6\}$, we can deduce that $\underline{\sigma}_5$ and $\underline{\sigma}_6$ are invariant under complex conjugation. It follows from $\underline{\lambda}_5(b_{19}) \uparrow^{\tilde{S}_{17}} = -2\sigma_6 + \sigma_9 + \sigma_{10}$ that $\sigma_9 - x_5\underline{\sigma}_5 - x_6\underline{\sigma}_6$ and $\sigma_{10} - y_5\underline{\sigma}_5 - y_6\underline{\sigma}_6$ are irreducible Brauer characters for suitable $x_5, y_5, x_6, y_6 \in \{0, 1, 2\}$ such that $x_5 + y_5 = 2$ and $x_6 + y_6 = 2$. As $\sigma_9^{\circledast} = \sigma_{10}$, clearly $x_5 = x_6 = y_5 = y_6 = 1$. Thus $\underline{\sigma}_i := \sigma_i - \sigma_6$ is irreducible for $i \in \{9, 10\}$. We have $\underline{\Lambda}_2(b_{19}) \uparrow^{\tilde{S}_{17}} = \Sigma_6 = \sigma_5^* + 2\sigma_6^*$ and $\deg \sigma_5^* = 0$. Hence $\underline{\Sigma}_5 := \sigma_5^* + \sigma_6^*$ and $\underline{\Sigma}_6 := \sigma_6^*$ are projective indecomposable characters. By Table 17.7, $\underline{\Sigma}_i := \Sigma_i$ is indecomposable for $i \in \{7, 8\}$. Moreover, it follows from $\underline{\Lambda}_5(b_{19}) \uparrow^{\tilde{S}_{17}} = -\underline{\Sigma}_7 - \underline{\Sigma}_8 + \Sigma_9 + \Sigma_{10}$ that $\underline{\Sigma}_9 := \Sigma_9 - \underline{\Sigma}_7$

and $\underline{\Sigma}_{10} := \Sigma_{10} - \underline{\Sigma}_8$ are indecomposable.

Table 17.6: Some characters for B_{29}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma\langle\langle 14, 3, +\rangle\rangle\vert_{5'}$	56320	$\sigma(17) \otimes \Sigma[12, 3, 2] \,\vert_{B_{29}}$	19584000
2	$\sigma\langle\langle 14, 3, -\rangle\rangle\vert_{5'}$	56320	$\Sigma[\![5^2, 4, + : 3]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	208640000
3	$\sigma\langle\langle 13, 4, +\rangle\rangle\vert_{5'}$	161280	$\Sigma[\![5, 4 : 5, 3, -]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	142304000
4	$\sigma\langle\langle 13, 4, -\rangle\rangle\vert_{5'}$	161280	$\Sigma[\![5^2, 3 : 4, -]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	161856000
5	$\sigma\langle\langle 9, 8, -\rangle\rangle\vert_{5'}$	183040	$\Sigma[\![9, 4 : 4]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	18880000
6	$\underline{\lambda}_2(b_{19})\!\uparrow^{\tilde{S}_{17}}$	156160	$\Sigma[\![5^2, 4, 2]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	26304000
7	$\sigma[\![8, 4, 1 : 4, +]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	1541120	$\Sigma[\![9, 4, - : 3, 1]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	11968000
8	$\sigma[\![9, 4, 1, - : 3]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	1541120	$\Sigma[\![9, 4, + : 3, 1]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	11968000
9	$\sigma\langle\langle 8, 4, 3, 2, +\rangle\rangle\vert_{5'}$	1697280	$\Sigma[\![8, 4, 3 : 2, -]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	26112000
10	$\sigma\langle\langle 8, 4, 3, 2, -\rangle\rangle\vert_{5'}$	1697280	$\Sigma[\![8, 4, 3 : 2, +]\!]\!\uparrow^{\tilde{S}_{17}} \vert_{B_{29}}$	26112000
11	$\sigma\langle\langle 9, 4, 3, 1, +\rangle\rangle\vert_{5'}$	2872320		
12	$\sigma\langle\langle 9, 4, 3, 1, -\rangle\rangle\vert_{5'}$	2872320		

As $\underline{\sigma}_i$ and $\underline{\Sigma}_i$ correspond for $i \in \{9, 10\}$, we can improve Σ_2 by $\Sigma_{2a} := \Sigma_2 - 2\underline{\Sigma}_9 - 2\underline{\Sigma}_{10}$ and Σ_4 by $\Sigma_{4a} := \Sigma_4 - \underline{\Sigma}_9 - \underline{\Sigma}_{10}$, see Table 17.8. Furthermore, we have

$$\begin{aligned} \underline{\lambda}_3(b_{19})\!\uparrow^{\tilde{S}_{17}} &= -\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_3 + \sigma_4, \\ \sigma_{11} &= \underline{\sigma}_1 + \underline{\sigma}_2 - \sigma_3 - \sigma_4 + \sigma_7 + \underline{\sigma}_{10}, \\ \sigma_{12} &= \underline{\sigma}_1 + \underline{\sigma}_2 - \sigma_3 - \sigma_4 + \sigma_8 + \underline{\sigma}_9. \end{aligned}$$

These relations imply that we may exchange σ_i with $\sigma_{ia} := \sigma_i - \sigma_3 - \sigma_4 + \underline{\sigma}_1 + \underline{\sigma}_2$ for $i \in \{7, 8\}$. Then $\underline{\sigma}_7 := \sigma_{7a}$ and $\underline{\sigma}_8 := \sigma_{8a}$ are irreducible since $\underline{\lambda}_4(b_{19})\!\uparrow^{\tilde{S}_{17}}\!= \sigma_{7a} + \sigma_{8a}$. The pairs $(\underline{\sigma}_i, \underline{\Sigma}_i)$ for $i \in \{7, 8\}$ yield $\Sigma_{2b} := \Sigma_{2a} - 5\underline{\Sigma}_7 - 5\underline{\Sigma}_8$ and $\Sigma_{4b} := \Sigma_{4a} - 3\underline{\Sigma}_7 - 3\underline{\Sigma}_8$. We have $\underline{\Lambda}_1(b_{19})\!\uparrow^{\tilde{S}_{17}}\!= \Sigma_1$. By consideration of degrees, it follows that the indecomposable constituents of Σ_1 are either $\sigma_1^* + \sigma_3^*$ and $\sigma_2^* + \sigma_4^*$, or $\sigma_1^* + \sigma_4^*$ and $\sigma_2^* + \sigma_3^*$. Table 17.9 and the relations $\Sigma_1 \downarrow_{\tilde{A}_{17}}\!= 2\underline{\Lambda}_1(b_{19})$ and $\Sigma_{4b} \downarrow_{\tilde{A}_{17}}\!= 4\underline{\Lambda}_1(b_{19}) + \underline{\Lambda}_2(b_{19}) + \underline{\Lambda}_3(b_{19})$ show that Σ_{4b} contains Σ_1 twice. Thus we may swap Σ_{4b} and $\Sigma_{4c} := \Sigma_{4b} - 2\Sigma_1$. There remain two independent problems: we need to decide whether $\sigma_3 - \underline{\sigma}_1$ and $\sigma_4 - \underline{\sigma}_2$, or $\sigma_3 - \underline{\sigma}_2$ and $\sigma_4 - \underline{\sigma}_1$ are irreducible Brauer characters, and we need to split σ_6 into its irreducible constituents $\underline{\sigma}_5$ and $\underline{\sigma}_6$. We use condensation and the trace formula to resolve this.

Table 17.7: Mutual scalar products (B_{29})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}
σ_1	1	.	2	2	.	.	.	.	.	.
σ_2	1	1	2	2	.	.	.	.	.	.
σ_3	1	1	4	2	1	.	.	.	.	.
σ_4	1	1	4	3	1	.	.	.	.	.
σ_5	.	1	3	1	1	1	.	.	.	.
σ_6	.	1	1	1	.	2	.	.	.	.
σ_7	.	6	5	4	2	.	1	.	1	.
σ_8	.	6	5	4	2	.	.	1	.	1
σ_9	.	3	2	2	.	2	.	.	1	.
σ_{10}	.	3	2	2	.	2	.	.	.	1

Table 17.8: Mutual scalar products (B_{29})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
$\underline{\sigma}_1$	1	.	.	2	.	.	.	.	.	.
$\underline{\sigma}_2$	1	1	.	2	.	.	.	.	.	.
σ_3	1	1	1	2	.	.	.	.	.	.
σ_4	1	1	1	3	.	.	.	.	.	.
σ_5	.	1	1	1	1	.	.	.	.	.
σ_6	.	1	.	1	1	1	.	.	.	.
σ_7	.	6	2	4	.	.	1	.	.	.
σ_8	.	6	2	4	.	.	.	1	.	.
$\underline{\sigma}_9$	.	2	.	1	.	.	.	.	1	.
$\underline{\sigma}_{10}$	.	2	.	1	.	.	.	.	.	1

Table 17.9: Mutual scalar products (B_{29})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_{2b}	Σ_3	Σ_{4b}	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$
$\underline{\sigma}_1$	1	.	.	2	.	.
$\underline{\sigma}_2$	1	1	.	2	.	.
σ_3	1	1	1	2	.	.
σ_4	1	1	1	3	.	.
σ_5	.	1	1	1	1	.
σ_6	.	1	.	1	1	1

The condensation subgroup is $K := \langle t_{(7,1^{10})}, t_{(1^7,7,1^3)}, zt_{(1^{14},3)}, \tilde{s}_1 \rangle \cong (7^2 \times 3) : 2$ where $s_1 := (1,8)(2,9)(3,10)(4,11)(5,12)(6,13)(7,14)(16,17)$. Let $256a^-$ be a basic spin module of $\tilde{S}_{17}$ over $F := GF(5)$, and let $16a^+$ be the reduced permutation module of S_{17}. We construct an $F\tilde{S}_{17}$-module $1700a^+$ by chopping the sequence $(16a^+ \otimes 16a^+, 16a^+ \otimes 103a^+, 16a^+ \otimes 543a^+)$. Consider $V := 256a^- \otimes 1700a^+$ with Brauer character $\sigma(17) \otimes \sigma[13,4] = \underline{\sigma}_1 + \underline{\sigma}_2 + \sigma_3 + \sigma_4$. We have $\langle \underline{\sigma}_i \downarrow_K, 1_K \rangle = 128$ for $i \in \{1,2\}$ and $\langle \sigma_j \downarrow_K, 1_K \rangle = 384$ for $j \in \{3,4\}$. Using $a := t_1$, $b := t_{16} \cdots t_1$, and $c := (ab)^3 b$ we define further elements $d_1 := ((ab)^2 bab^2 cb)^3 b$ and $d_2 := d_1 c$. For the trace formula we need

$$x_1 := (1,13,8,3,14,9,4,16,11,6)(2,15,10,5,17,12,7).$$

Denote by e the trivial idempotent of K. With respect to the condensation algebra $C(\tilde{x}) := \langle ed_1e, ed_2e, e\tilde{x}_1e \rangle$, the condensed module Ve has two composition factors of dimension 256 that correspond to the irreducible Brauer characters we are looking for. The traces $\pm\zeta_5$ of $e\tilde{x}_1e$ on these constituents determine $\underline{\sigma}_3 := \sigma_3 - \underline{\sigma}_1$ and $\underline{\sigma}_4 := \sigma_4 - \underline{\sigma}_2$ as irreducible Brauer characters. Now $\Sigma_1 = \underline{\sigma}_1^* + \underline{\sigma}_2^*$ can be split into its indecomposable constituents $\underline{\Sigma}_1 := \underline{\sigma}_1^*$ and $\underline{\Sigma}_2 := \underline{\sigma}_2^*$. Exchanging these with Σ_1 and Σ_2 in the basic set of projective characters, we get $\sigma_6 = \Sigma_4^* + \Sigma_5^* + \Sigma_6^*$. In order to split σ_6, we condense $W := 256a^- \otimes 1597a^+$ over $C(\tilde{x}_2)$ where

$$x_2 := (1,7,12)(2,6,11,17,5,10,15,3,8,13,16,4,9,14).$$

Embarking from the reduced S_{13}-permutation module $12a^+$ over F, we chop the sequence $(12a^+ \otimes 12a^+, 65a^+ \otimes 65a^+, 12a^+ \otimes 428a^+, 12a^+ \otimes 144a^+)$. This gives an FS_{13}-module $233a^+$ with Brauer character $\sigma[7,6]$. Chopping the sequence $(233a^+ \uparrow^{S_{14}}, 233b^+ \uparrow^{S_{15}}, 610a^+ \uparrow^{S_{16}}, 610b^+ \uparrow^{S_{17}})$ gives an FS_{17}-module $1597a^+$ that affords the Brauer character $\sigma[9,8]$. Then We has two distinct $C(\tilde{x}_2)$-constituents of dimension 296 corresponding to $\underline{\sigma}_5$ and $\underline{\sigma}_6$. We identify $\underline{\sigma}_5 = \Sigma_4^* + \Sigma_5^*$ and $\underline{\sigma}_6 = \Sigma_6^*$ by means of the traces of $e\tilde{x}_2e$. The decomposition matrix of B_{29} is Table 17.19.

17.6 The block B_{28}

Basic sets of Brauer characters σ_i and projective characters $\underline{\Sigma}_i$ for $i \in \{1, \ldots, 10\}$ are defined by Table 17.10 where $\Sigma_0 := \sigma\langle\langle 8,5,3,1,+\rangle\rangle + \sigma\langle\langle 8,5,3,1,-\rangle\rangle + \sigma\langle\langle 8,6,3\rangle\rangle$ is a projective indecomposable character of the block B_{30} of defect 1, see Figure A.27 in

Appendix A. We see in Table 17.11 that $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1, \dots, 10\} \setminus \{7\}$. As $\sigma_{11} = 2\underline{\sigma}_1 - \underline{\sigma}_3 + 3\underline{\sigma}_5 + \sigma_7$, it is clear that $\underline{\sigma}_7 := \sigma_7 - \underline{\sigma}_3$ is irreducible too. The decomposition matrix of B_{28} is Table 17.20.

Table 17.10: Some characters for B_{28}

i	σ_i	$\deg \sigma_i$	$\underline{\Sigma}_i$	$\deg \underline{\Sigma}_i$
1	$\sigma(17)$	256	$\Sigma[\![5^3, 1, -]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	97824000
2	$\sigma[\![6, 5, 4, 1, +]\!]\uparrow^{\tilde{S}_{17}}$	13056	$\Sigma[\![7, 5, 4]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	39168000
3	$\sigma[\![7, 5, 3 : 2]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	43520	$\Sigma[\![7, 6, 2 : 2, +]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	11968000
4	$\sigma[\![5^2, 3, 2, 1, +]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	126208	$\Sigma[\![5^2, 4, 2]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	51488000
5	$\sigma[\![9, 6, 1, -]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	312832	$\underline{\Sigma}_5(B_{27}) \otimes \sigma[16, 1] \vert_{B_{28}}/2$	22464000
6	$\sigma[\![6, 4, 3, 2, 1, +]\!]\uparrow^{\tilde{S}_{17}}$	426496	$\Sigma[\![7, 4, 3, 2]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	54400000
7	$\sigma\langle\!\langle 12, 3, 2\rangle\!\rangle\vert_{5'}$	678912	$\Sigma[\![8, 6, 2, -]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}/2$	14144000
8	$\sigma\langle\!\langle 6, 5, 4, 2, -\rangle\!\rangle\vert_{5'}$	792064	$\Sigma[\![6, 5, 2, 1 : 2, 1]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}/2$	21216000
9	$\sigma[\![7, 5, 3, 1, -]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	1017856	$\Sigma_0 \otimes \sigma[16, 1] \vert_{B_{28}}/2$	27200000
10	$\sigma[\![8, 5, 2, 1, -]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	1610240	$\Sigma[\![8, 4, 2 : 2, 1]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}/4$	29376000
11	$\sigma[\![9, 5, 2]\!]\uparrow^{\tilde{S}_{17}} \vert_{B_{28}}$	1574400		

17.7 The block b_{18}

Basic sets of Brauer characters λ_i and projective characters Λ_i for $i \in \{1, \dots, 20\}$ are defined by Table 17.12. Recall that $\underline{\lambda}_1 := \lambda_1$ and $\underline{\lambda}_2 := \lambda_2$ are irreducible. First $\underline{\sigma}_3(B_{28})\downarrow_{\tilde{A}_{17}} = \lambda_5 + \lambda_6$ shows that $\underline{\lambda}_5 := \lambda_5$ and $\underline{\lambda}_6 := \lambda_6$ are irreducible. Consider

$$\underline{\sigma}_2(B_{28})\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 + \lambda_3 + \lambda_4, \tag{i}$$

$$\underline{\sigma}_7(B_{28})\downarrow_{\tilde{A}_{17}} = -\underline{\lambda}_5 - \underline{\lambda}_6 + \lambda_9 + \lambda_{10}, \tag{ii}$$

$$\underline{\sigma}_4(B_{28})\downarrow_{\tilde{A}_{17}} = \lambda_7, \tag{iii}$$

$$\underline{\sigma}_6(B_{28})\downarrow_{\tilde{A}_{17}} = -2\lambda_7 + \lambda_{11} + \lambda_{12}, \tag{iv}$$

$$\underline{\sigma}_8(B_{28})\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - 3\lambda_7 + \lambda_{13} + \lambda_{14}. \tag{v}$$

From (i) and Table 17.13 we can deduce that $\underline{\lambda}_i := \lambda_i - \underline{\lambda}_1 - \underline{\lambda}_2$ is an irreducible Brauer character for $i \in \{3, 4\}$. Similarly, (ii) yields $\underline{\lambda}_9 := \lambda_9 - \underline{\lambda}_6$ and $\underline{\lambda}_{10} := \lambda_{10} - \underline{\lambda}_5$, both are irreducible. By (iii) and the decomposition $\lambda_7 = \Lambda_7^* + \Lambda_8^*$, the Brauer atoms $\underline{\lambda}_7 := \Lambda_7^*$

Table 17.11: Mutual scalar products (B_{28})

$\langle\cdot,\cdot\rangle$	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}_4$	$\underline{\Sigma}_5$	$\underline{\Sigma}_6$	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$
σ_1	1	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.	.
σ_4	.	.	.	1	.	.	.	.	.	.
σ_5	.	.	.	.	1	.	.	.	.	.
σ_6	.	.	.	.	.	1	.	.	.	.
σ_7	.	.	1	.	.	.	1	.	.	.
σ_8	.	.	.	.	.	.	.	1	.	.
σ_9	.	.	.	.	.	.	.	.	1	.
σ_{10}	.	.	.	.	.	.	.	.	.	1

Table 17.12: Some characters for b_{18}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(17)$	128	$\lambda(17)^c \otimes \Lambda[14, 1^3] \mid_{b_{18}}$	130128000
2	$\lambda(17)^c$	128	$\lambda(17) \otimes \Lambda[14, 1^3] \mid_{b_{18}}$	130128000
3	$\sigma\llbracket 5^2, 4, 1, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	6784	$\Lambda\llbracket 7, 5, 4, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	19584000
4	$\sigma\llbracket 5^2, 4, 1, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	6784	$\Lambda\llbracket 7, 5, 4, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	19584000
5	$\sigma\llbracket 7, 5, 3, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	21760	$\Lambda\llbracket 7, 6, 3\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	11968000
6	$\sigma\llbracket 7, 5, 3, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	21760	$\Lambda\llbracket 7, 5, 2, + : 3, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	39168000
7	$\lambda\llbracket 5^2, 3, 2, 1\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	126208	$\Lambda\llbracket 5^2, 4, 2, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	25744000
8	$\lambda\llbracket 5, 4, 1, + : 5, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	559232	$\Lambda\llbracket 5^2, 4, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	25744000
9	$\lambda\langle\langle 12, 3, 2, -\rangle\rangle\mid_{5'}$	339456	$\Lambda\llbracket 8, 6, 2\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}/2$	14144000
10	$\lambda\langle\langle 12, 3, 2, +\rangle\rangle\mid_{5'}$	339456	$\Sigma\llbracket 8, 5, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	98976000
11	$\lambda\langle\langle 7, 4, 3, 2, 1, +\rangle\rangle\mid_{5'}$	339456	$\Lambda\llbracket 7, 4, 3, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	27200000
12	$\sigma\llbracket 5, 4, 3, 2, 1, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	339456	$\Lambda\llbracket 7, 4, 3, 2, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	27200000
13	$\lambda\llbracket 5^2, 4, 2, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	585600	$\Sigma\llbracket 6, 5, 4, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	76432000
14	$\lambda\llbracket 5^2, 4, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	585600	$\Sigma\llbracket 6, 5, 4, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	76432000
15	$\lambda\llbracket 9, 6, 1\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	312832	$\Lambda\llbracket 9, 5, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	25920000
16	$\lambda\langle\langle 12, 4, 1, -\rangle\rangle\mid_{5'}$	502656	$\Lambda\llbracket 9, 5, 2, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	25920000
17	$\lambda\langle\langle 7, 6, 4, +\rangle\rangle\mid_{5'}$	1096704	$\Lambda\llbracket 7, 5, 2, 1, + : 2\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	27200000
18	$\lambda\llbracket 7, 5, 3, 1\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	1017856	$\Lambda\llbracket 6, 4, + : 5, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	738320000
19	$\lambda\langle\langle 11, 4, 2, +\rangle\rangle\mid_{5'}$	1279488	$\lambda\langle\langle 11, 6, 2, 1, +\rangle\rangle\mid_{5'}\downarrow_{\tilde{A}_{17}} \mid_{b_{18}}/3$	29376000
20	$\lambda\langle\langle 11, 4, 2, -\rangle\rangle\mid_{5'}$	1279488	$\Sigma\llbracket 9, 5, 1, -\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	59616000
21	$\sigma\llbracket 5^2, 3, 2, +\rrbracket\uparrow^{\tilde{A}_{17}} \mid_{b_{18}}$	1949952	$\lambda(17) \otimes \Lambda[13, 1^4] \mid_{b_{18}}$	114896000
22			$\lambda(17)^c \otimes \Lambda[13, 1^4] \mid_{b_{18}}$	114896000
23			$\lambda(17) \otimes \Lambda[15, 1^2] \mid_{b_{18}}$	70128000
24			$\lambda(17)^c \otimes \Lambda[9, 2^4, -] \mid_{b_{18}}$	70128000

Table 17.13: Mutual scalar products ($\mathfrak{b}_{18}$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}	Λ_{19}	Λ_{20}
λ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	1	1	1	.	.	1	.	.	.	.	.	.	2	1	.	.	.	1	.	.
λ_4	1	1	.	1	.	.	.	.	.	.	.	.	1	2	.	.	.	2	.	.
λ_5	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.
λ_6	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.
λ_7	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_8	6	6	2	2	.	2	1	.	.	5	.	.	6	6	1	2	.	29	.	6
λ_9	.	.	.	.	1	1	.	.	1	2	.	.	1	.	.	.	.	11	.	.
λ_{10}	.	.	.	.	1	.	.	.	1	1	.	.	.	1	.	.	.	11	.	.
λ_{11}	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.
λ_{12}	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.
λ_{13}	1	1	.	.	.	.	2	1	.	.	.	.	1	.	.	.	.	.	.	.
λ_{14}	1	1	.	.	.	.	1	2	.	.	.	.	.	1	.	.	.	1	.	.
λ_{15}	2	2	.	.	.	.	.	.	.	3	.	.	.	.	1	1	.	15	.	4
λ_{16}	2	2	.	1	1	1	.	.	1	3	.	.	2	2	1	.	.	20	.	2
λ_{17}	1	1	1	1	1	2	1	1	.	.	1	1	3	3	.	.	1	6	.	.
λ_{18}	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	2	4	.	.
λ_{19}	4	4	.	.	.	.	.	.	1	4	.	.	.	1	1	1	.	27	1	3
λ_{20}	4	4	.	.	.	.	.	.	1	5	.	.	1	.	1	1	.	27	1	2

and $\underline{\lambda}_8 := \Lambda_8^*$ are irreducible Brauer characters. Then (iv), (v) and Table 17.13 lead to the irreducible Brauer characters

$$\begin{aligned}
\underline{\lambda}_{11} &:= \lambda_{11} - \underline{\lambda}_7 - \underline{\lambda}_8, \\
\underline{\lambda}_{12} &:= \lambda_{12} - \underline{\lambda}_7 - \underline{\lambda}_8 \quad , \\
\underline{\lambda}_{13} &:= \lambda_{13} - \underline{\lambda}_1 - \underline{\lambda}_2 - 2\underline{\lambda}_7 - \underline{\lambda}_8, \\
\underline{\lambda}_{14} &:= \lambda_{14} - \underline{\lambda}_1 - \underline{\lambda}_2 - \underline{\lambda}_7 - 2\underline{\lambda}_8.
\end{aligned}$$

We have $\underline{\Sigma}_9(B_{28}) \downarrow_{\tilde{A}_{17}} = \Lambda_{17} = \lambda_{17}^* + 2\lambda_{18}^*$ and $\underline{\Sigma}_{10}(B_{28}) \downarrow_{\tilde{A}_{17}} = \Lambda_{19} = \lambda_{19}^* + \lambda_{20}^*$. Since $\deg \lambda_{17}^* = 0$, we get the projective indecomposable characters $\underline{\Lambda}_{17} := \lambda_{17}^* + \lambda_{18}^*$, $\underline{\Lambda}_{18} := \lambda_{18}^*$, and $\underline{\Lambda}_i := \lambda_i^*$ for $i \in \{19, 20\}$. The relations $\underline{\Sigma}_i(B_{28})\downarrow_{\tilde{A}_{17}} = \Lambda_{2i-1} + \Lambda_{2i}$ for $i \in \{2, 4, 6\}$ show that $\underline{\Lambda}_j := \Lambda_j$ is indecomposable for $j \in \{3, 4, 7, 8, 11, 12\}$. Next, we have

$$\underline{\sigma}_{10}(B_{28})\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - \underline{\lambda}_9 - \underline{\lambda}_{10} - \lambda_{15} + \lambda_{19} + \lambda_{20}.$$

From Table 17.13 (see columns 13, 14) we can deduce that $\underline{\lambda}_9$ is contained in λ_{20}, and $\underline{\lambda}_{10}$ is contained in λ_{19}. Thus $\lambda_{19a} := \lambda_{19} - \underline{\lambda}_{10}$ and $\lambda_{20a} := \lambda_{20} - \underline{\lambda}_9$ improve λ_{19} and λ_{20}. The relation

$$\underline{\Sigma}_8(B_{28})\downarrow_{\tilde{A}_{17}} = -3\underline{\Lambda}_3 - 3\underline{\Lambda}_4 - \Lambda_9 + \Lambda_{13} + \Lambda_{14}, \tag{vi}$$

gives $\Lambda_{13a} := \Lambda_{13} - 2\underline{\Lambda}_3 - \underline{\Lambda}_4$ and $\Lambda_{14a} := \Lambda_{14} - \underline{\Lambda}_3 - 2\underline{\Lambda}_4$. Now $\Lambda_9 = \underline{\lambda}_9^* + \underline{\lambda}_{10}^* + \lambda_{16}^*$ is the sum of two associate, projective indecomposable characters since $\underline{\Sigma}_9(B_{28})\downarrow_{\tilde{A}_{17}} = \Lambda_9$. By (vi) and Table 17.14 it is now clear that one of them is contained in Λ_{13a} and the other one is contained in Λ_{14a}. Hence these are $\underline{\Lambda}_9 := \lambda_9^* + \lambda_{16}^*$ and $\underline{\Lambda}_{10} := \lambda_{10}^*$. It follows that $\underline{\Lambda}_{13} := \Lambda_{13a} - \underline{\Lambda}_9$ and $\underline{\Lambda}_{14} := \Lambda_{14a} - \underline{\Lambda}_{10}$ are indecomposable.

Since $\underline{\lambda}_i$ and $\underline{\Lambda}_i$ correspond for $i \in \{3, 4, 7, \ldots, 14\}$, we read off Table 17.14 the improvements $\Lambda_{6a} := \Lambda_6 - \underline{\Lambda}_3$, $\lambda_{16a} := \lambda_{16} - \underline{\lambda}_4 - \underline{\lambda}_9$, and $\lambda_{17a} := \lambda_{17} - \underline{\lambda}_3 - \underline{\lambda}_4 - \underline{\lambda}_7 - \underline{\lambda}_8 - \underline{\lambda}_{11} - \underline{\lambda}_{12}$. The relations $\Lambda_i = \Lambda_{i-20} + \underline{\Lambda}_9 + \underline{\Lambda}_{10} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ for $i \in \{21, 22\}$ yield $\Lambda_{ja} := \Lambda_j - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ for $j \in \{1, 2\}$. Moreover, we have $\Lambda_i = \Lambda_{ia-22} + \underline{\Lambda}_{13} + \underline{\Lambda}_{14} - \Lambda_{15} - \Lambda_{16}$ for $i \in \{23, 24\}$ and $\underline{\Sigma}_5(B_{28})\downarrow_{\tilde{A}_{17}} = \Lambda_{15} + \Lambda_{16} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$. We obtain $\Lambda_{ib} := \Lambda_{ia} - \Lambda_{15} - \Lambda_{16} + \underline{\Lambda}_{19} + \underline{\Lambda}_{20}$ in exchange with Λ_{ia} for $i \in \{1, 2\}$. Decomposing $\Lambda_i = \Lambda_{ib-22} + \underline{\Lambda}_{13} + \underline{\Lambda}_{14} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ again for $i \in \{23, 24\}$ gives $\underline{\Lambda}_j := \Lambda_{jb} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ for $j \in \{1, 2\}$. Then $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$ are indecomposable since their sum equals $\underline{\Sigma}_1(B_{28})\downarrow_{\tilde{A}_{17}}$.

These improvements of the projective characters lead to $\underline{\sigma}_5(B_{28})\downarrow_{\tilde{A}_{17}} = \lambda_{15} = \Lambda_{15}^* + \Lambda_{16}^*$. Thus $\underline{\lambda}_{15} := \Lambda_{15}^*$ and $\underline{\lambda}_{16} := \Lambda_{16}^*$ are irreducible Brauer characters.

Table 17.14: Mutual scalar products ($\mathfrak{b}_{18}$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	$\underline{\Lambda}_3$	$\underline{\Lambda}_4$	Λ_5	Λ_6	$\underline{\Lambda}_7$	$\underline{\Lambda}_8$	Λ_9	Λ_{10}	$\underline{\Lambda}_{11}$	$\underline{\Lambda}_{12}$	Λ_{13a}	Λ_{14a}	Λ_{15}	Λ_{16}	$\underline{\Lambda}_{17}$	$\underline{\Lambda}_{18}$	$\underline{\Lambda}_{19}$	$\underline{\Lambda}_{20}$
$\underline{\lambda}_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_2$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_3$	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_5$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_6$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_7$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_8$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_9$	.	.	.	.	.	.	.	.	1	2	.	.	1	.	.	.	.	.	.	.
$\underline{\lambda}_{10}$	.	.	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.
$\underline{\lambda}_{11}$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\underline{\lambda}_{12}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$\underline{\lambda}_{13}$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$\underline{\lambda}_{14}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
λ_{15}	2	2	.	.	.	.	.	.	.	3	.	.	.	.	1	1	.	.	.	.
λ_{16}	2	2	.	1	1	1	.	.	1	3	.	.	1	.	1	.	.	.	.	.
λ_{17}	1	1	1	1	1	2	1	1	.	.	1	1	.	.	.	.	1	.	.	.
λ_{18}	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	1	.	.
λ_{19a}	4	4	.	.	.	.	.	.	.	3	.	.	.	.	1	1	.	.	1	.
λ_{20a}	4	4	.	.	.	.	.	.	.	3	.	.	.	.	1	1	.	.	.	1

The relation $\lambda_{21} = 2\underline{\lambda}_1 + 2\underline{\lambda}_2 - \underline{\lambda}_6 + 2\underline{\lambda}_7 + 2\underline{\lambda}_8 + \underline{\lambda}_{13} + 2\underline{\lambda}_{14} + \lambda_{17a}$ yields $\lambda_{17a} := \lambda_{17} - \underline{\lambda}_6$. As $\underline{\lambda}_i$ corresponds to $\underline{\Lambda}_i$ for $i \in \{1, 2\}$, we can improve λ_{ia} by $\lambda_{ib} := \lambda_{ia} - \underline{\lambda}_1 - \underline{\lambda}_2$ for $i \in \{17, 19, 20\}$. Now $\underline{\lambda}_{17} := \lambda_{17b} = \Sigma^*_{17b}$ is a Brauer atom, hence irreducible, and $\underline{\lambda}_{18} := \underline{\lambda}^{\circledast}_{17} = \underline{\lambda}^{c}_{17}$ is irreducible too. We are left with $\underline{\sigma}_{10}(B_{28})\downarrow_{\tilde{A}_{17}} = -\underline{\lambda}_{15} - \underline{\lambda}_{16} + \lambda_{19b} + \lambda_{20b}$ and $\underline{\Sigma}_5(B_{28})\downarrow_{\tilde{A}_{17}} = \Lambda_{15} + \Lambda_{16} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$. The corresponding mutual scalar products are given in Table 17.15.

Table 17.15: Mutual scalar products (b_{18})

$\langle\cdot,\cdot\rangle$	Λ_{15}	Λ_{16}	$\underline{\Lambda}_{19}$	$\underline{\Lambda}_{20}$
$\underline{\lambda}_{15}$	1	.	.	.
$\underline{\lambda}_{16}$	.	1	.	.
λ_{19b}	1	1	1	.
λ_{20b}	1	1	.	1

Recall the objects used for condensation in 17.5. We find an FS_{17}-module $7410a^+$ as a constituent of $16a^+ \otimes 1700a^+$ whose restriction to A_{17} affords $\lambda[12, 4, 1]$. Moreover, set $F_1 := \mathrm{GF}(25)$ and denote by $1920a^-$ a constituent of $256a^- \otimes 16a^+$ that affords $\sigma(16, 1)$ or $\sigma(16, 1)^c$. We condense the $F_1\tilde{A}_{17}$-module $U := (1920a^- \otimes 7410a^+)\downarrow_{\tilde{A}_{17}}$ with respect to the trivial idempotent f of $L := \langle K, \tilde{s}_2\rangle$ with $s_2 := (2, 3, 5)(4, 7, 6)(9, 10, 12)(11, 14, 13)$ and $L \cong (7^2 \times 3) : (2 \times 3)$. Let $D := \langle fd_1f, fd_3f, f\tilde{x}_3f\rangle$ where $d_3 := ad_2$ and

$$x_3 := (1, 14, 10, 6)(2, 15, 11, 7, 17, 13, 9, 5)(3, 16, 12, 8, 4).$$

Then the traces $\pm\zeta_{25}^9$ of $f\tilde{x}_3f$ on the two distinct 826-dimensional D-composition factors of Uf determine $\underline{\lambda}_{19} := \lambda_{19b} - \underline{\lambda}_{16}$ and $\underline{\lambda}_{20} := \lambda_{20b} - \underline{\lambda}_{15}$ as irreducible Brauer characters, and we are done. The decomposition matrix of b_{18} is Table 17.21.

Table 17.16: 5-modular decomposition matrix of block b_{17} of $\tilde{A}_{17}$
(defect 2, bar core $(6,1)$, weight 2, content $(8,6,3)$)

			$[\![6,5,3,2,1]\!]$ 282880	$[\![6,5^2,1]\!]$ 1920	$[\![7,6,3,1]\!]$ 2302720	$[\![8,6,2,1]\!]$ 522240	$[\![10,6,1]\!]$ 465920
⋆	1920	$\langle\!\langle 16,1\rangle\!\rangle$	.	1	.	.	.
⋆	465920	$\langle\!\langle 11,6\rangle\!\rangle$	.	.	.	.	1
	990080	$\langle\!\langle 11,5,1,+\rangle\!\rangle$	.	1	.	1	1
	990080	$\langle\!\langle 11,5,1,-\rangle\!\rangle$	.	1	.	1	1
⋆	522240	$\langle\!\langle 11,3,2,1\rangle\!\rangle$	.	.	.	1	.
	1272960	$\langle\!\langle 10,6,1,+\rangle\!\rangle$	1	1	.	1	1
	1272960	$\langle\!\langle 10,6,1,-\rangle\!\rangle$	1	1	.	1	1
	3111680	$\langle\!\langle 8,6,2,1\rangle\!\rangle$	1	2	1	1	.
⋆	2872320	$\langle\!\langle 7,6,3,1\rangle\!\rangle$	2	2	1	.	.
⋆	282880	$\langle\!\langle 6,5,3,2,1,+\rangle\!\rangle$	1	.	.	.	.
	282880	$\langle\!\langle 6,5,3,2,1,-\rangle\!\rangle$	1	.	.	.	.

Table 17.17: 5-modular decomposition matrix of block B_{27} of $\tilde{S}_{17}$
(defect 2, bar core $(6,1)$, weight 2, content $(8,6,3)$)

			$[\![6,5,3,2,1,+]\!]$ 282880	$[\![6,5,3,2,1,-]\!]$ 282880	$[\![6,5^2,1,+]\!]$ 1920	$[\![6,5^2,1,-]\!]$ 1920	$[\![7,6,3,1,+]\!]$ 2302720	$[\![7,6,3,1,-]\!]$ 2302720	$[\![8,6,2,1,+]\!]$ 522240	$[\![8,6,2,1,-]\!]$ 522240	$[\![10,6,1,+]\!]$ 465920	$[\![10,6,1,-]\!]$ 465920
$\star$	1920	$\langle\!\langle 16,1,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
$\star$	1920	$\langle\!\langle 16,1,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.
$\star$	465920	$\langle\!\langle 11,6,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.
$\star$	465920	$\langle\!\langle 11,6,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1
	1980160	$\langle\!\langle 11,5,1\rangle\!\rangle$	.	.	1	1	.	.	1	1	1	1
$\star$	522240	$\langle\!\langle 11,3,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.
$\star$	522240	$\langle\!\langle 11,3,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.
	2545920	$\langle\!\langle 10,6,1\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	1
$\star$	3111680	$\langle\!\langle 8,6,2,1,+\rangle\!\rangle$	.	1	1	1	1	.	1	.	.	.
	3111680	$\langle\!\langle 8,6,2,1,-\rangle\!\rangle$	1	.	1	1	.	1	.	1	.	.
$\star$	2872320	$\langle\!\langle 7,6,3,1,+\rangle\!\rangle$	1	1	1	1	.	1	.	.	.	.
$\star$	2872320	$\langle\!\langle 7,6,3,1,-\rangle\!\rangle$	1	1	1	1	1	.	.	.	.	.
$\star$	565760	$\langle\!\langle 6,5,3,2,1\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.

Table 17.18: 5-modular decomposition matrix of block $\mathfrak{b}_{19}$ of $\tilde{A}_{17}$
(defect 2, bar core $(4,3)$, weight 2, content $(6,7,4)$)

			$[\![5^2,4,3]\!]$ 78080	$[\![8,4,3,2]\!]$ 1541120	$[\![8,5,4]\!]$ 56320	$[\![9,4,3,1]\!]$ 1331200	$[\![9,5,3]\!]$ 104960
⋆	56320	$\langle\!\langle 14,3\rangle\!\rangle$	.	.	1	.	.
⋆	161280	$\langle\!\langle 13,4\rangle\!\rangle$	.	.	1	.	1
⋆	1436160	$\langle\!\langle 10,4,3,+\rangle\!\rangle$	.	.	.	1	1
	1436160	$\langle\!\langle 10,4,3,-\rangle\!\rangle$	.	.	.	1	1
⋆	183040	$\langle\!\langle 9,8\rangle\!\rangle$	1	.	.	.	1
	3111680	$\langle\!\langle 9,5,3,+\rangle\!\rangle$	1	1	1	1	1
	3111680	$\langle\!\langle 9,5,3,-\rangle\!\rangle$	1	1	1	1	1
	2872320	$\langle\!\langle 9,4,3,1\rangle\!\rangle$	.	1	.	1	.
⋆	1675520	$\langle\!\langle 8,5,4,+\rangle\!\rangle$	1	1	1	.	.
	1675520	$\langle\!\langle 8,5,4,-\rangle\!\rangle$	1	1	1	.	.
	1697280	$\langle\!\langle 8,4,3,2\rangle\!\rangle$	2	1	.	.	.

Table 17.19: 5-modular decomposition matrix of block B_{29} of $\tilde{\mathrm{S}}_{17}$
(defect 2, bar core $(4,3)$, weight 2, content $(6,7,4)$)

			$[\![5^2,4,3,+]\!]$ 78080	$[\![5^2,4,3,-]\!]$ 78080	$[\![8,4,3,2,+]\!]$ 1541120	$[\![8,4,3,2,-]\!]$ 1541120	$[\![8,5,4,+]\!]$ 56320	$[\![8,5,4,-]\!]$ 56320	$[\![9,4,3,1,+]\!]$ 1331200	$[\![9,4,3,1,-]\!]$ 1331200	$[\![9,5,3,+]\!]$ 104960	$[\![9,5,3,-]\!]$ 104960
$\star$	56320	$\langle\!\langle 14,3,+\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	.
$\star$	56320	$\langle\!\langle 14,3,-\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	.
$\star$	161280	$\langle\!\langle 13,4,+\rangle\!\rangle$	.	.	.	.	1	.	.	.	1	.
$\star$	161280	$\langle\!\langle 13,4,-\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	1
	2872320	$\langle\!\langle 10,4,3\rangle\!\rangle$	.	.	.	.	.	.	1	1	1	1
$\star$	183040	$\langle\!\langle 9,8,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	1
$\star$	183040	$\langle\!\langle 9,8,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	1	.
	6223360	$\langle\!\langle 9,5,3\rangle\!\rangle$	1	1	1	1	1	1	1	1	1	1
$\star$	2872320	$\langle\!\langle 9,4,3,1,+\rangle\!\rangle$	.	.	1	.	.	.	.	1	.	.
$\star$	2872320	$\langle\!\langle 9,4,3,1,-\rangle\!\rangle$	.	.	.	1	.	.	1	.	.	.
	3351040	$\langle\!\langle 8,5,4\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.
$\star$	1697280	$\langle\!\langle 8,4,3,2,+\rangle\!\rangle$	1	1	.	1	.	.	.	.	.	.
$\star$	1697280	$\langle\!\langle 8,4,3,2,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	.	.

Table 17.20: 5-modular decomposition matrix of block B_{28} of $\tilde{S}_{17}$
(defect 3, bar core (2), weight 3, content $(7,7,3)$)

			$[\![5^2,4,2,1]\!]$ 126208	$[\![5^3,2]\!]$ 256	$[\![6,5,4,2]\!]$ 792064	$[\![7,4,3,2,1]\!]$ 426496	$[\![7,5,3,2]\!]$ 1017856	$[\![7,5,4,1]\!]$ 13056	$[\![7,6,4]\!]$ 43520	$[\![8,7,2]\!]$ 635392	$[\![9,5,2,1]\!]$ 1610240	$[\![9,6,2]\!]$ 312832
$\star$	256	$\langle\!\langle 17\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
$\star$	13312	$\langle\!\langle 15,2,+\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	.
	13312	$\langle\!\langle 15,2,-\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	.
$\star$	56576	$\langle\!\langle 14,2,1\rangle\!\rangle$	.	.	.	.	.	1	1	.	.	.
$\star$	326144	$\langle\!\langle 12,5,+\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	1
	326144	$\langle\!\langle 12,5,-\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	1
	1005312	$\langle\!\langle 12,4,1\rangle\!\rangle$	.	2	.	.	.	1	1	1	.	1
$\star$	678912	$\langle\!\langle 12,3,2\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.
	2558976	$\langle\!\langle 11,4,2\rangle\!\rangle$	.	2	.	.	.	.	.	1	1	1
$\star$	439296	$\langle\!\langle 10,7,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	439296	$\langle\!\langle 10,7,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	4978688	$\langle\!\langle 10,5,2\rangle\!\rangle$	2	4	.	2	.	2	.	.	2	2
$\star$	2036736	$\langle\!\langle 10,4,2,1,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.
	2036736	$\langle\!\langle 10,4,2,1,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.
	1867008	$\langle\!\langle 9,7,1\rangle\!\rangle$	1	2	1	.	.	.	.	1	.	1
	5544448	$\langle\!\langle 9,6,2\rangle\!\rangle$	2	4	1	2	1	2	1	1	1	1
	3620864	$\langle\!\langle 9,5,2,1,+\rangle\!\rangle$	1	2	.	2	1	1	.	.	1	.
	3620864	$\langle\!\langle 9,5,2,1,-\rangle\!\rangle$	1	2	.	2	1	1	.	.	1	.
	2489344	$\langle\!\langle 8,7,2\rangle\!\rangle$	.	2	1	.	1	.	1	1	.	.
$\star$	2193408	$\langle\!\langle 7,6,4\rangle\!\rangle$	2	2	.	2	1	2	1	.	.	.
	2376192	$\langle\!\langle 7,5,4,1,+\rangle\!\rangle$	1	2	1	1	1	1	.	.	.	.
	2376192	$\langle\!\langle 7,5,4,1,-\rangle\!\rangle$	1	2	1	1	1	1	.	.	.	.
	2489344	$\langle\!\langle 7,5,3,2,+\rangle\!\rangle$	2	2	1	1	1	.	.	.	.	.
	2489344	$\langle\!\langle 7,5,3,2,-\rangle\!\rangle$	2	2	1	1	1	.	.	.	.	.
$\star$	678912	$\langle\!\langle 7,4,3,2,1\rangle\!\rangle$	2	.	.	1	.	.	.	.	.	.
$\star$	792064	$\langle\!\langle 6,5,4,2,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
	792064	$\langle\!\langle 6,5,4,2,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.

Table 17.21: 5-modular decomposition matrix of block $\mathfrak{b}_{18}$ of $\tilde{A}_{17}$
(defect 3, bar core (2), weight 3, content $(7,7,3)$)

			$[\![5^2,4,2,1,+]\!]$ 63104	$[\![5^2,4,2,1,-]\!]$ 63104	$[\![5^3,2,+]\!]$ 128	$[\![5^3,2,-]\!]$ 128	$[\![6,5,4,2,+]\!]$ 396032	$[\![6,5,4,2,-]\!]$ 396032	$[\![7,4,3,2,1,+]\!]$ 213248	$[\![7,4,3,2,1,-]\!]$ 213248	$[\![7,5,3,2,+]\!]$ 508928	$[\![7,5,3,2,-]\!]$ 508928	$[\![7,5,4,1,+]\!]$ 6528	$[\![7,5,4,1,-]\!]$ 6528	$[\![7,6,4,+]\!]$ 21760	$[\![7,6,4,-]\!]$ 21760	$[\![8,7,2,+]\!]$ 317696	$[\![8,7,2,-]\!]$ 317696	$[\![9,5,2,1,+]\!]$ 805120	$[\![9,5,2,1,-]\!]$ 805120	$[\![9,6,2,+]\!]$ 156416	$[\![9,6,2,-]\!]$ 156416
⋆	128	$\langle\!\langle 17,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	128	$\langle\!\langle 17,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	13312	$\langle\!\langle 15,2\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
⋆	28288	$\langle\!\langle 14,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
⋆	28288	$\langle\!\langle 14,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.
⋆	326144	$\langle\!\langle 12,5\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	1	1
⋆	502656	$\langle\!\langle 12,4,1,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	1	1	.	.	1	.	.	1	.
	502656	$\langle\!\langle 12,4,1,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	1
⋆	339456	$\langle\!\langle 12,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	339456	$\langle\!\langle 12,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
⋆	1279488	$\langle\!\langle 11,4,2,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.
⋆	1279488	$\langle\!\langle 11,4,2,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1
⋆	439296	$\langle\!\langle 10,7\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
	2489344	$\langle\!\langle 10,5,2,+\rangle\!\rangle$	1	1	2	2	.	.	1	1	.	.	1	1	.	.	.	.	1	1	1	1
	2489344	$\langle\!\langle 10,5,2,-\rangle\!\rangle$	1	1	2	2	.	.	1	1	.	.	1	1	.	.	.	.	1	1	1	1
	2036736	$\langle\!\langle 10,4,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.
⋆	933504	$\langle\!\langle 9,7,1,+\rangle\!\rangle$	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.
	933504	$\langle\!\langle 9,7,1,-\rangle\!\rangle$	1	.	1	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	1
⋆	2772224	$\langle\!\langle 9,6,2,+\rangle\!\rangle$	1	1	2	2	.	1	1	1	1	.	1	1	.	1	1	.	1	.	.	1
	2772224	$\langle\!\langle 9,6,2,-\rangle\!\rangle$	1	1	2	2	1	.	1	1	.	1	1	1	1	.	.	1	.	1	1	.
	3620864	$\langle\!\langle 9,5,2,1\rangle\!\rangle$	1	1	2	2	.	.	2	2	1	1	1	1	.	.	.	.	1	1	.	.
⋆	1244672	$\langle\!\langle 8,7,2,+\rangle\!\rangle$	.	.	1	1	1	.	.	.	.	1	.	.	1	.	.	1	.	.	.	.
	1244672	$\langle\!\langle 8,7,2,-\rangle\!\rangle$	.	.	1	1	.	1	.	.	1	.	.	.	.	1	1	.	.	.	.	.
⋆	1096704	$\langle\!\langle 7,6,4,+\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	.	1	1	.	1	.	.	.	.	.	.
⋆	1096704	$\langle\!\langle 7,6,4,-\rangle\!\rangle$	1	1	1	1	.	.	1	1	.	1	1	1	1	.	.	.	.	.	.	.
	2376192	$\langle\!\langle 7,5,4,1\rangle\!\rangle$	1	1	2	2	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.
	2489344	$\langle\!\langle 7,5,3,2\rangle\!\rangle$	2	2	2	2	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.
⋆	339456	$\langle\!\langle 7,4,3,2,1,+\rangle\!\rangle$	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	339456	$\langle\!\langle 7,4,3,2,1,-\rangle\!\rangle$	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	792064	$\langle\!\langle 6,5,4,2\rangle\!\rangle$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.

18 The 7-modular spin characters of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

Both $\tilde{S}_{17}$ and $\tilde{A}_{17}$ have six spin 7-blocks of defect 0. The constants of the spin 7-blocks of non-zero defect are given in Table 18.1.

Table 18.1: Spin 7-blocks of non-zero defect of $\tilde{S}_{17}$ and $\tilde{A}_{17}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
70	17	9	2	(3)	2	38	22	18
71	7	6	1	$(6,4)$	1	39	5	3
72	22	18	2	$(2,1)$	2	40	17	9
73	5	3	1	$(5,4,1)$	1	41	7	6

The 7-modular basic spin character of $\tilde{S}_{17}$ is $\sigma(17) := \sigma\langle\langle 17\rangle\rangle|_{7'}$. Its restriction to $\tilde{A}_{17}$ is the sum of $\lambda(17) := \lambda\langle\langle 17,+\rangle\rangle|_{7'}$ and $\lambda(17)^c = \lambda\langle\langle 17,-\rangle\rangle|_{7'}$. Two more irreducible Brauer characters of $\tilde{S}_{17}$ are $\sigma(16,1) := \sigma\langle\langle 16,1,+\rangle\rangle|_{7'}$ and $\sigma(16,1)^a = \sigma\langle\langle 16,1,-\rangle\rangle|_{7'}$ which restrict irreducibly to $\lambda(16,1) := \lambda\langle\langle 16,1\rangle\rangle|_{7'}$.

18.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{71}, B_{73}, b_{39}, and b_{41} are given in Appendix A as Figures A.29, A.30, A.31, and A.32, respectively.

18.2 The block B_{70}

$\Sigma_0 := \sigma\langle\langle 8,5,4\rangle\rangle + \sigma\langle\langle 11,5,1\rangle\rangle$ is a projective indecomposable character of B_{73}, see Figure A.30 in Appendix A, that appears in the definition of Σ_7 in Table 18.2. The basic sets $\{\sigma_1,\dots,\sigma_9\}$ and $\{\Sigma_1,\dots,\Sigma_9\}$ defined by Table 18.2 satisfy $\langle\sigma_i,\Sigma_j\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,9\}$ such that $(i,j) \neq (9,5)$. Hence $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1,\dots,8\}$. Since $\langle\sigma_9,\Sigma_5\rangle = 1$ and $\sigma_{10} = 2\underline{\sigma}_1 + \underline{\sigma}_4 - \underline{\sigma}_5 + 2\underline{\sigma}_7 + \sigma_9$, we see that $\mathrm{IBr}(B_{70})$ is completed by $\underline{\sigma}_9 := \sigma_9 - \underline{\sigma}_5$. The decomposition matrix of B_{70} is Table 18.10.

Table 18.2: Some characters for B_{70}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(17)$	256	$\Sigma[\![7^2, 2, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	17398528
2	$\sigma[\![9, 6, 1, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	56064	$\Sigma[\![9, 6, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	3863552
3	$\sigma[\![9, 5, 2, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	270336	$\Sigma[\![8, 2, 1 : 6, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}/2$	8529920
4	$\sigma[\![7, 6, 2, 1, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	382976	$\Sigma[\![7, 6, 2 : 2]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}/2$	20484352
5	$\sigma[\![10, 4, 1 : 2, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	408576	$\Sigma[\![10, 4, 2, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	8529920
6	$\sigma\langle\!\langle 6, 5, 3, 2, 1\rangle\!\rangle\vert_{7'}$	565760	$\Sigma[\![6, 5, 3 : 2, 1]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	22604288
7	$\sigma[\![8, 6, 2, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	1836032	$\sigma[16, 1] \otimes \Sigma_0 \vert_{B_{70}}/2$	19405568
8	$\sigma[\![7, 4, 3, 2, +]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	1923584	$\sigma[16, 1] \otimes \sigma\langle\!\langle 8, 5, 3, 1, +\rangle\!\rangle \vert_{B_{70}}$	22604288
9	$\sigma\langle\!\langle 10, 4, 3\rangle\!\rangle\vert_{7'}$	2872320	$\sigma(17) \otimes \Sigma[9, 4^2] \vert_{B_{70}}/2$	14074368
10	$\sigma[\![7, 4, 3 : 2, 1, -]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{70}}$	6519296		

18.3 The block b_{38}

We have a basic set of Brauer characters λ_i and a basic set of projective characters Λ_i for $i \in \{1, \ldots, 18\}$ defined by Table 18.4. We know that $\underline{\lambda}_1 := \lambda_1$ and $\underline{\lambda}_2 := \lambda_2$ are irreducible. Since $\underline{\sigma}_5(B_{70})\downarrow_{\tilde{A}_{17}} = \lambda_3 + \lambda_4$ and $\underline{\sigma}_6(B_{70})\downarrow_{\tilde{A}_{17}} = \lambda_7 + \lambda_8$, the same holds for $\underline{\lambda}_i := \lambda_i$, $i \in \{3, 4, 7, 8\}$. By Table 18.3, we have $\underline{\sigma}_8(B_{70})\downarrow_{\tilde{A}_{17}} = \lambda_{15} = \Lambda^*_{15} + \Lambda^*_{16}$. Hence the Brauer atoms $\underline{\lambda}_{15} := \Lambda^*_{15}$ and $\underline{\lambda}_{16} := \Lambda^*_{16}$ are irreducible Brauer characters. From $\underline{\sigma}_2(B_{70})\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 + \lambda_{11} + \lambda_{12}$ and Table 18.3 we see that $\underline{\lambda}_i := \lambda_i - \underline{\lambda}_1 - \underline{\lambda}_2$ is irreducible for $i \in \{11, 12\}$. Similarly, $\underline{\sigma}_9(B_{70})\downarrow_{\tilde{A}_{17}} = -\underline{\lambda}_3 - \underline{\lambda}_4 + \lambda_5 + \lambda_6$ implies that $\underline{\lambda}_5 := \lambda_5 - \underline{\lambda}_3$ and $\underline{\lambda}_6 := \lambda_6 - \underline{\lambda}_4$ are irreducible. Moreover, $\underline{\Sigma}_2(B_{70})\downarrow_{\tilde{A}_{17}} = \Lambda_{11} + \Lambda_{12}$ and $\underline{\Sigma}_4(B_{70})\downarrow_{\tilde{A}_{17}} = \Lambda_{17} + \Lambda_{18}$ show that $\underline{\Lambda}_i := \Lambda_i$ is indecomposable for $i \in \{11, 12, 17, 18\}$. Then $\underline{\Sigma}_5(B_{70})\downarrow_{\tilde{A}_{17}} = \Lambda_3 + \Lambda_4 - \Lambda_9 - \underline{\Lambda}_{11} - \underline{\Lambda}_{12}$ and the decompositions in terms of the projective atoms give $\Lambda_{3a} := \Lambda_3 - \underline{\Lambda}_{12}$ and $\Lambda_{4a} := \Lambda_4 - \underline{\Lambda}_{11}$. Now the relation $\underline{\sigma}_3(B_{70})\downarrow_{\tilde{A}_{17}} = \lambda_9 + \lambda_{10} - \underline{\lambda}_{11} - \underline{\lambda}_{12}$ and the decompositions in terms of the Brauer atoms yield $\underline{\lambda}_9 := \lambda_9 - \underline{\lambda}_{12}$ and $\underline{\lambda}_{10} := \lambda_{10} - \underline{\lambda}_{11}$, both irreducible. Instead of λ_{13} and λ_{14} we obtain $\lambda_{13a} := \lambda_{13} - \underline{\lambda}_1 - \underline{\lambda}_2 - \underline{\lambda}_{10} - \underline{\lambda}_{11}$ and $\lambda_{14a} := \lambda_{14} - \underline{\lambda}_1 - \underline{\lambda}_2 - \underline{\lambda}_9 - \underline{\lambda}_{12}$, respectively, as $\underline{\sigma}_7(B_{70})\downarrow_{\tilde{A}_{17}} = -2\underline{\lambda}_1 - 2\underline{\lambda}_2 - \underline{\lambda}_9 - \underline{\lambda}_{10} - \underline{\lambda}_{11} - \underline{\lambda}_{12} + \lambda_{13} + \lambda_{14} - \lambda_{18}$. Some mutual scalar products at this point are given in Table 18.5.

Since $\underline{\lambda}_i$ corresponds to $\underline{\Lambda}_i$ for $i \in \{11, 12\}$, we can improve the basic set of projective characters by $\Lambda_{5a} := \Lambda_5 - \underline{\Lambda}_{11} - 2\underline{\Lambda}_{12}$, $\Lambda_{6a} := \Lambda_6 - 2\underline{\Lambda}_{11} - \underline{\Lambda}_{12}$, $\Lambda_{10a} := \Lambda_{10} - \underline{\Lambda}_{12}$, and $\Lambda_{13a} := \Lambda_{13} - 2\underline{\Lambda}_{11} - 2\underline{\Lambda}_{12}$. As $\underline{\Lambda}_{10} := \Lambda_{10a}$ is a projective atom it is indecomposable.

Table 18.3: Mutual scalar products ($\mathfrak{b}_{38}$)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}
λ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_4	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_5	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_6	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_7	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
λ_8	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
λ_9	.	.	1	1	3	3	.	.	1	1	.	1	3	.	.	.	.	.
λ_{10}	.	.	1	1	3	3	.	.	1	1	1	.	2	.	.	.	.	.
λ_{11}	1	1	.	1	1	2	.	.	.	.	1	.	2	.	.	.	.	.
λ_{12}	1	1	1	.	2	1	.	.	.	1	.	1	2	.	.	.	.	.
λ_{13}	1	1	1	1	3	3	.	.	1	1	1	.	3	1	1	1	1	.
λ_{14}	1	1	1	1	3	3	.	.	1	1	.	1	3	1	1	1	.	1
λ_{15}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.
λ_{16}	1	1	.	.	.	.	1	1	.	.	.	.	1	1	2	2	1	1
λ_{17}	.	.	1	1	2	2	.	.	1	.	.	.	1	1	.	1	.	.
λ_{18}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1

Then $\underline{\Lambda}_9 := \underline{\Lambda}_{10}^{\circledast} = \Lambda_9 - \underline{\Lambda}_{10}$ is indecomposable too. The pairs $(\underline{\lambda}_i, \underline{\Lambda}_i)$ for $i \in \{9, 10\}$ allow further improvements $\Lambda_{3b} := \Lambda_{3b} - \underline{\Lambda}_{10}$, $\Lambda_{4b} := \Lambda_{4b} - \underline{\Lambda}_9$, $\Lambda_{5b} := \Lambda_{5a} - \underline{\Lambda}_9 - 2\underline{\Lambda}_{10}$, $\Lambda_{6b} := \Lambda_{6a} - 2\underline{\Lambda}_9 - \underline{\Lambda}_{10}$, $\Lambda_{13b} := \Lambda_{13a} - \underline{\Lambda}_9$. Then Λ_{13b} is a projective atom, thus $\underline{\Lambda}_{13} := \Lambda_{13b}$ and $\underline{\Lambda}_{14} := \underline{\Lambda}_{13}^{\circledast} = \Lambda_{14} - \underline{\Lambda}_{13}$ are indecomposable. It becomes clear from $\underline{\Sigma}_8(B_{70})\downarrow_{\tilde{A}_{17}} = -\underline{\Lambda}_{13} - \underline{\Lambda}_{14} + \Lambda_{15} - \Lambda_{16} - \underline{\Lambda}_{17} - \underline{\Lambda}_{18}$ and the decompositions in terms of the projective atoms that $\lambda_{15}^* = \Lambda_{15} - \underline{\Lambda}_{13} - \underline{\Lambda}_{18}$ and $\lambda_{16}^* = \Lambda_{16} - \underline{\Lambda}_{14} - \underline{\Lambda}_{17}$ are projective indecomposable characters. This results in $\lambda_{18} = \Lambda_{17}^* + \Lambda_{18}^*$. As $\lambda_{18} = \underline{\sigma}_4(B_{70})\downarrow_{\tilde{A}_{17}}$, the Brauer atoms $\underline{\lambda}_{17} := \Lambda_{17}^*$ and $\underline{\lambda}_{18} := \Lambda_{18}^*$ are irreducible Brauer characters. Finally, it follows from $\underline{\sigma}_7(B_{70})\downarrow_{\tilde{A}_{17}} = \lambda_{13a} + \lambda_{14a} - \underline{\lambda}_{17} - \underline{\lambda}_{18}$ that $\underline{\lambda}_{13} := \lambda_{13a} - \underline{\lambda}_{17}$ and $\underline{\lambda}_{14} := \lambda_{14a} - \underline{\lambda}_{18}$ are elements of $\mathrm{IBr}(b_{38})$. The decomposition matrix of b_{38} is Table 18.11.

Table 18.4: Some characters for b_{38}

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(17)$	128	$\lambda(17) \otimes \Lambda[13, 1^4] \mid_{b_{38}}$	8699264
2	$\lambda(17)^c$	128	$\lambda(17) \otimes \Lambda[17] \mid_{b_{38}}$	8699264
3	$\lambda[\![10, 4, + : 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	204288	$\Lambda[\![10, 4, + : 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	10461696
4	$\lambda[\![10, 4, + : 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	204288	$\Lambda[\![10, 4, + : 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	10461696
5	$\lambda\langle\!\langle 10, 4, 3, -\rangle\!\rangle\mid_{7'}$	1436160	$\Lambda[\![9, 5, + : 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	25627392
6	$\lambda\langle\!\langle 10, 4, 3, +\rangle\!\rangle\mid_{7'}$	1436160	$\Lambda[\![9, 5, + : 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	25627392
7	$\lambda\langle\!\langle 6, 5, 3, 2, 1, -\rangle\!\rangle\mid_{7'}$	282880	$\Lambda[\![6, 5, 3, 2, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	11302144
8	$\lambda\langle\!\langle 6, 5, 3, 2, 1, +\rangle\!\rangle\mid_{7'}$	282880	$\Lambda[\![6, 5, 3, 2, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	11302144
9	$\lambda\langle\!\langle 13, 3, 1, +\rangle\!\rangle\mid_{7'}$	163200	$\Lambda[\![10, 5, 1]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	8529920
10	$\lambda\langle\!\langle 13, 3, 1, -\rangle\!\rangle\mid_{7'}$	163200	$\Sigma[\![10, 5, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	6196736
11	$\lambda[\![7, 6, 1, + : 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	28288	$\Lambda[\![10, 6, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	1931776
12	$\lambda[\![7, 6, 1, + : 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	28288	$\Lambda[\![10, 6, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	1931776
13	$\lambda\langle\!\langle 10, 6, 1, +\rangle\!\rangle\mid_{7'}$	1272960	$\Lambda[\![8, 6, + : 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	21694848
14	$\lambda\langle\!\langle 10, 6, 1, -\rangle\!\rangle\mid_{7'}$	1272960	$\Lambda[\![8, 5, 3]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	19405568
15	$\underline{\sigma}_8(B_{70})\downarrow_{\tilde{A}_{17}}$	1923584	$\Sigma[\![7, 5, 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	31247104
16	$\lambda\langle\!\langle 8, 6, 3, -\rangle\!\rangle\mid_{7'}$	2828800	$\Sigma[\![7, 5, 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	31247104
17	$\lambda\langle\!\langle 10, 5, 2, +\rangle\!\rangle\mid_{7'}$	2489344	$\Lambda[\![7, 6, 3, -]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	10242176
18	$\underline{\sigma}_8(B_{70})\downarrow_{\tilde{A}_{17}}$	382976	$\Lambda[\![7, 6, 3, +]\!]\uparrow^{\tilde{A}_{17}} \mid_{b_{38}}$	10242176

Table 18.5: Mutual scalar products (b_{38})

$\langle\cdot,\cdot\rangle$	Λ_{3a}	Λ_{4a}	Λ_5	Λ_6	Λ_9	Λ_{10}	$\underline{\Lambda}_{11}$	$\underline{\Lambda}_{12}$	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	$\underline{\Lambda}_{17}$	$\underline{\Lambda}_{18}$
$\underline{\lambda}_9$	.	1	1	2	1	.	.	.	1	.	.	.	.	.
$\underline{\lambda}_{10}$	1	.	2	1	1	1	.	.	.	.	.	.	.	.
$\underline{\lambda}_{11}$	.	.	1	2	.	.	1	.	2	.	.	.	.	.
$\underline{\lambda}_{12}$	.	.	2	1	.	1	.	1	2	.	.	.	.	.
λ_{13a}	.	.	.	.	.	.	.	.	1	1	1	1	1	.
λ_{14a}	.	.	.	.	.	.	.	.	.	1	1	1	.	1
$\underline{\lambda}_{15}$	.	.	.	.	.	.	.	.	.	.	1	.	.	.
$\underline{\lambda}_{16}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.
λ_{17}	1	1	2	2	1	.	.	.	1	1	.	1	.	.
λ_{18}	.	.	.	.	.	.	.	.	.	.	1	1	1	1

18.4 The block b_{40}

The Brauer characters $\underline{\lambda}_i$ and the projective characters $\underline{\Lambda}_j$ defined by Table 18.6 satisfy $\langle\underline{\lambda}_i, \underline{\Lambda}_j\rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, 9\}$. The decomposition matrix of b_{40} is Table 18.12.

Table 18.6: Some characters for b_{40}

i	$\underline{\lambda}_i$	$\deg\underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg\underline{\Lambda}_i$
1	$\lambda(16,1)$	1920	$\lambda(17)\otimes\Lambda[11,1^6]\mid_{b_{40}}$	4371584
2	$\lambda(17)\otimes\lambda[14,3]\mid_{b_{40}}$	11392	$\Lambda[\![9,6:2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	2119936
3	$\lambda[\![9,5,2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	16896	$\Lambda[\![9,5,1,+:2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	8103424
4	$\lambda(17)^c\otimes\lambda[9,8]\mid_{b_{40}}$	171648	$\Lambda[\![7,6,2:2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	10762752
5	$\lambda\langle\!\langle 7,4,3,2,1,-\rangle\!\rangle\mid_{7'}$	339456	$\Lambda[\![6,4,3,2,1]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	11088896
6	$\lambda[\![10,4,+:2,1]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	505344	$\Lambda[\![10,3,2,+:2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	2558976
7	$\lambda[\![8,6,2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	731648	$\Lambda[\![8,5,1:2,1]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	11088896
8	$\lambda[\![7,4,3,2]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	1357824	$\Lambda[\![6,3,2,+:3,2,1]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}/2$	11302144
9	$\lambda[\![7,4,3,+:2,1]\!]\uparrow^{\tilde{A}_{17}}\mid_{b_{40}}$	1514496	$\lambda(17)\otimes\Lambda[8,4^2,1]\mid_{b_{40}}/2$	8529920

18.5 The block B_{72}

Basic sets of Brauer characters σ_i and projective characters Σ_i for $i \in \{1, \ldots, 18\}$ are defined by Table 18.7. Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$. Since $\underline{\lambda}_i(b_{40})\uparrow^{\tilde{S}_{17}} = \sigma_{2i-1} + \sigma_{2i}$ for $i \in \{3, 7, 8, 9\}$ and $\underline{\lambda}_6(b_{40})\uparrow^{\tilde{S}_{17}} = \sigma_9 + \sigma_{10}$, we have clearly $\underline{\sigma}_i := \sigma_i \in \mathrm{IBr}(B_{72})$ for

$i \in \{5,6,9,10,13,14,15,16,17,18\}$. As $\underline{\lambda}_5(b_{40})\uparrow^{\tilde{S}_{17}} = \sigma_{11} = \Sigma_{11}^* + \Sigma_{11}^*$, see Table 18.8, also $\underline{\sigma}_{11} := \Sigma_{11}^*$ and $\underline{\sigma}_{12} := \Sigma_{12}^*$ are irreducible Brauer characters.

Table 18.7: Some characters for B_{72}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(16,1)^a$	1920	$\underline{\Lambda}_1(b_{40})\uparrow^{\tilde{S}_{17}}$	8743168
2	$\sigma(16,1)$	1920	$\Sigma[\![7^2,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	17254272
3	$\sigma\langle\!\langle 15,2,+\rangle\!\rangle\vert_{7'}$	13312	$\underline{\Lambda}_2(b_{40})\uparrow^{\tilde{S}_{17}}$	4239872
4	$\sigma\langle\!\langle 15,2,-\rangle\!\rangle\vert_{7'}$	13312	$\Sigma[\![7^2,1:2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	30369024
5	$\sigma[\![9,5,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	16896	$\Sigma[\![9,5,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	8103424
6	$\sigma[\![9,5,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	16896	$\Sigma[\![9,5:2,1,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	8103424
7	$\sigma\langle\!\langle 9,8,-\rangle\!\rangle\vert_{7'}$	183040	$\Sigma[\![7,2:7,1,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	80231424
8	$\sigma\langle\!\langle 9,8,+\rangle\!\rangle\vert_{7'}$	183040	$\Sigma[\![7^2,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	17254272
9	$\sigma[\![10,4,1:2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	505344	$\Sigma[\![10,4:2,1,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	2558976
10	$\sigma[\![10,4:2,1,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	505344	$\Sigma[\![10,4,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	2558976
11	$\underline{\lambda}_5(b_{40})\uparrow^{\tilde{S}_{17}}$	678912	$\Sigma[\![7,4,3,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	22391040
12	$\sigma\langle\!\langle 8,4,3,2,+\rangle\!\rangle\vert_{7'}$	1697280	$\Sigma[\![7,4,3,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	22391040
13	$\sigma[\![8,6,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	731648	$\Sigma[\![8,6,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	11088896
14	$\sigma[\![8,6,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	731648	$\Sigma[\![8,6,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	11088896
15	$\sigma[\![7,4,3,2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	1357824	$\Sigma[\![8,4,3:2,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	30920960
16	$\sigma[\![7,4,3,2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	1357824	$\Sigma[\![8,4,3:2,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	30920960
17	$\sigma[\![9,4,3,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	1514496	$\Sigma[\![9,4,3,-]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	8529920
18	$\sigma[\![9,4,3,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	1514496	$\Sigma[\![9,4,3,+]\!]\uparrow^{\tilde{S}_{17}}\vert_{B_{72}}$	8529920

Table 18.8 shows that $\underline{\Sigma}_i := \Sigma_i \in \mathrm{IBr}(B_{72})$ for $i \in \{9,10,13,14,17,18\}$. It follows from $\underline{\Lambda}_8(b_{40})\uparrow^{\tilde{S}_{17}} = -\underline{\Sigma}_{13} - \underline{\Sigma}_{14} + \Sigma_{15} + \Sigma_{16} - \underline{\Sigma}_{17} - \underline{\Sigma}_{18}$ that both $\underline{\Sigma}_{15} := \Sigma_{15} - \underline{\Sigma}_{14} - \underline{\Sigma}_{17}$ and $\underline{\Sigma}_{16} := \Sigma_{16} - \underline{\Sigma}_{13} - \underline{\Sigma}_{18}$ are indecomposable. In addition, $\underline{\Lambda}_4(b_{40})\uparrow^{\tilde{S}_{17}} = -\Sigma_1 + \Sigma_2 - \Sigma_3 + \Sigma_8$ shows that $\underline{\Lambda}_4(b_{40})$ is the sum of the atoms $\underline{\Sigma}_7 := \sigma_7^*$ and $\underline{\Sigma}_8 := \sigma_8^*$ that are projective indecomposable characters. Further projective indecomposables, $\underline{\Sigma}_{11} := \Sigma_{11} - \underline{\Sigma}_{16}$ and $\Sigma_{12} - \underline{\Sigma}_{15}$, can be deduced from $\underline{\Lambda}_5(b_{40})\uparrow^{\tilde{S}_{17}} = \Sigma_{11} + \Sigma_{12} - \underline{\Sigma}_{15} - \underline{\Sigma}_{16}$. By definition, we know that $\Sigma_1 = \sigma_1^* + \ldots + \sigma_4^*$ is the sum of two associate projective indecomposable characters. Since $\deg \sigma_1^* = \deg \sigma_2^* > 0$ and $\deg \sigma_3^* = \deg \sigma_4^* < 0$ these indecomposables have the form $\sigma_1^* + \sigma_i^*$ and $\sigma_2^* + \sigma_j^*$ where $\{i,j\} = \{3,4\}$. Both of them restrict to $\underline{\Lambda}_1(b_{40})$. Therefore, the relation $\Sigma_4\downarrow_{\tilde{A}_{17}} = 4\underline{\Lambda}_1(b_{40}) + \underline{\Lambda}_2(b_{40}) + \underline{\Lambda}_4(b_{40})$ and the decomposition of Σ_4 in terms of the projective atoms imply that we may subtract Σ_1 twice from Σ_4. We

Table 18.8: Mutual scalar products (B_{72})

$\langle \cdot,\cdot \rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}
σ_1	1	.	.	2	.	.	2	1	.	.	.	.	.	.	.	.	.	.
σ_2	1	1	.	2	.	.	2	.	.	.	.	.	.	.	.	.	.	.
σ_3	1	1	1	2	.	.	4	1	.	.	.	.	.	.	.	.	.	.
σ_4	1	1	1	3	.	.	4	1	.	.	.	.	.	.	.	.	.	.
σ_5	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.
σ_6	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.
σ_7	.	1	1	1	.	.	3	1	.	.	.	.	.	.	.	.	.	.
σ_8	.	1	1	1	.	.	2	1	.	.	.	.	.	.	.	.	.	.
σ_9	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.
σ_{10}	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
σ_{11}	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
σ_{12}	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	1	.	.
σ_{13}	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.
σ_{14}	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	.
σ_{15}	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.
σ_{16}	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.
σ_{17}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.
σ_{18}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1

obtain $\Sigma_{4a} := \Sigma_4 - 2\Sigma_1$. Now we are left with the relations $\underline{\lambda}_2(b_{40})\uparrow^{\tilde{S}_{17}} = -\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_3 + \sigma_4$ and $\underline{\lambda}_4(b_{40})\uparrow^{\tilde{S}_{17}} = \underline{\sigma}_1 + \underline{\sigma}_2 - \sigma_3 - \sigma_4 + \sigma_7 + \sigma_8$ but the mutual scalar products given in Table 18.9 do not help to find the irreducible constituents of σ_3, σ_4, σ_7, or σ_8.

Table 18.9: Mutual scalar products (B_{72})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_{4a}	Σ_5	Σ_6	$\underline{\Sigma}_7$	$\underline{\Sigma}_8$
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.
$\underline{\sigma}_2$	1	1	.	.	.	.	.	.
σ_3	1	1	1	.	.	.	.	.
σ_4	1	1	1	1	.	.	.	.
σ_7	.	1	1	1	.	.	1	.
σ_8	.	1	1	1	.	.	.	1

Let $256a^-$ be a basic spin module of $\tilde{S}_{17}$ realized over $F := GF(7)$, and let $119a^+$ be the irreducible 119-dimensional constituent of the tensor product of the natural permutation module of S_{17} over F with itself. The Brauer character of $119a^+$ is $\sigma[15, 2]$. We condense the tensor product $V := 256a^- \otimes 119a^+$ with respect to the trivial idempotent e of the condensation subgroup $K := \langle t_{(9,1^8)}, zt_{(1^9,5,1^3)}\rangle \cong 9 \times 5$. For the trace formula we consider $x := (1, 15, 12, 9, 6, 3, 17, 14, 10, 7, 4)(2, 16, 13, 11, 8, 5)$. Let $a := t_1$, $b := t_{16} \cdots t_1$, $c := ((ab)^2(bab)^2abab^3)^2ab^2$, and $d := c(ab)^3b$. Our condensation algebra is $C(\tilde{x}, e) := \langle ece, ede, e\tilde{x}e\rangle$. We evaluate the trace of $e\tilde{x}e$ on the two 160-dimensional $C(\tilde{x}, e)$-composition factors of Ve that are in correspondence with the $F\tilde{S}_{17}$-modules affording the yet unknown irreducible constituents $\underline{\sigma}_3$ and $\underline{\sigma}_4$ of σ_3 and σ_4, respectively. These traces, ζ_7 and ζ_7^4, determine $\underline{\sigma}_3 = \sigma_3 - \underline{\sigma}_2$ and $\underline{\sigma}_4 = \sigma_4 - \underline{\sigma}_1$.
Similarly, we condense $W := 256a^- \otimes 4334a^+$ to determine the irreducible constituents $\underline{\sigma}_i \in \{\sigma_i - \underline{\sigma}_3, \sigma_i - \underline{\sigma}_4\}$ for $i \in \{7, 8\}$. Here $4334a^+$ affords $\sigma[9, 8]$ and we construct it as follows. Let $10a^+$ be the reduced permutation module of S_{11} over F. By chopping the sequence $(10a^+ \otimes 10a^+, 10a^+ \otimes 44a^+, 44a^+ \otimes 66a^+)$ we obtain an irreducible 131-dimensional FS_{11}-module $131a^+$. We get $4334a^+$ by chopping the sequence $(131a^+\uparrow^{S_{12}}, 131b^+\uparrow^{S_{13}}$, $417a^+\uparrow^{S_{14}}, 417b^+\uparrow^{S_{15}}, 1341a^+\uparrow^{S_{16}}, 1341b^+\uparrow^{S_{17}})$. We take $L := \langle K, zt_{(1^{14},3)}, \tilde{s}\rangle$ where $s := (2, 9)(3, 8)(4, 7)(5, 6)(11, 14)(12, 13)(16, 17)$, so $L \cong (9 \times 5 \times 3) : 2$. Let $f := e_L$ and $y := (1, 15, 11, 8, 5, 2, 14, 17, 13, 9, 4, 16, 12, 7)(3, 10, 6)$. The traces 0_F and 1_F of $f\tilde{y}f$ on the 632-dimensional $C(\tilde{y}, f)$-composition factors of Wf identify $\underline{\sigma}_7 = \sigma_7 - \underline{\sigma}_3$ and $\underline{\sigma}_8 = \sigma_8 - \underline{\sigma}_4$ as elements of $IBr(B_{72})$. The decomposition matrix of B_{72} is Table 18.13.

Table 18.10: 7-modular decomposition matrix of block B_{70} of $\tilde{S}_{17}$
(defect 2, bar core (3), weight 2, content $(5,5,5,2)$)

			$[\![6,5,3,2,1]\!]$ 565760	$[\![7,5,3,2]\!]$ 1923584	$[\![7,6,3,1]\!]$ 382976	$[\![7^2,3]\!]$ 256	$[\![8,6,3]\!]$ 1836032	$[\![9,5,3]\!]$ 2463744	$[\![10,4,3]\!]$ 408576	$[\![10,5,2]\!]$ 270336	$[\![10,6,1]\!]$ 56064
$\star$	256	$\langle\!\langle 17\rangle\!\rangle$	.	.	.	1	.	.	.	.	.
$\star$	56320	$\langle\!\langle 14,3,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	1
	56320	$\langle\!\langle 14,3,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	1
$\star$	326400	$\langle\!\langle 13,3,1\rangle\!\rangle$	.	.	.	.	.	.	.	1	1
$\star$	678912	$\langle\!\langle 12,3,2\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
$\star$	439296	$\langle\!\langle 10,7,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	1
	439296	$\langle\!\langle 10,7,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	1
$\star$	2545920	$\langle\!\langle 10,6,1\rangle\!\rangle$	.	.	1	2	1	.	.	1	1
	4978688	$\langle\!\langle 10,5,2\rangle\!\rangle$	.	.	.	.	1	1	1	1	.
$\star$	2872320	$\langle\!\langle 10,4,3\rangle\!\rangle$	.	.	.	.	.	1	1	.	.
	6223360	$\langle\!\langle 9,5,3\rangle\!\rangle$	.	1	.	.	1	1	.	.	.
	5657600	$\langle\!\langle 8,6,3\rangle\!\rangle$	2	1	2	2	1	.	.	.	.
	2872320	$\langle\!\langle 7,6,3,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
	2872320	$\langle\!\langle 7,6,3,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
$\star$	2489344	$\langle\!\langle 7,5,3,2,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
	2489344	$\langle\!\langle 7,5,3,2,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
$\star$	565760	$\langle\!\langle 6,5,3,2,1\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 18.11: 7-modular decomposition matrix of block b_{38} of $\tilde{\mathrm{A}}_{17}$
(defect 2, bar core (3), weight 2, content $(5,5,5,2)$)

			$[\![6,5,3,2,1,+]\!]$ 282880	$[\![6,5,3,2,1,-]\!]$ 282880	$[\![7,5,3,2,+]\!]$ 961792	$[\![7,5,3,2,-]\!]$ 961792	$[\![7,6,3,1,+]\!]$ 191488	$[\![7,6,3,1,-]\!]$ 191488	$[\![7^2,3,+]\!]$ 128	$[\![7^2,3,-]\!]$ 128	$[\![8,6,3,+]\!]$ 918016	$[\![8,6,3,-]\!]$ 918016	$[\![9,5,3,+]\!]$ 1231872	$[\![9,5,3,-]\!]$ 1231872	$[\![10,4,3,+]\!]$ 204288	$[\![10,4,3,-]\!]$ 204288	$[\![10,5,2,+]\!]$ 135168	$[\![10,5,2,-]\!]$ 135168	$[\![10,6,1,+]\!]$ 28032	$[\![10,6,1,-]\!]$ 28032
$\star$	128	$\langle\!\langle 17,+\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\star$	128	$\langle\!\langle 17,-\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\star$	56320	$\langle\!\langle 14,3\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1
$\star$	163200	$\langle\!\langle 13,3,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
$\star$	163200	$\langle\!\langle 13,3,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
$\star$	339456	$\langle\!\langle 12,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
$\star$	339456	$\langle\!\langle 12,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
$\star$	439296	$\langle\!\langle 10,7\rangle\!\rangle$	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	1	1
$\star$	1272960	$\langle\!\langle 10,6,1,+\rangle\!\rangle$	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	1	.
$\star$	1272960	$\langle\!\langle 10,6,1,-\rangle\!\rangle$	.	.	.	.	1	.	1	1	.	1	.	.	.	.	1	.	.	1
$\star$	2489344	$\langle\!\langle 10,5,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	1	.	.	1	1	.	.	.
	2489344	$\langle\!\langle 10,5,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	1	.	.
$\star$	1436160	$\langle\!\langle 10,4,3,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
$\star$	1436160	$\langle\!\langle 10,4,3,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
$\star$	3111680	$\langle\!\langle 9,5,3,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	1	1	.	.	.	.	.	.	.
	3111680	$\langle\!\langle 9,5,3,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.
$\star$	2828800	$\langle\!\langle 8,6,3,+\rangle\!\rangle$	1	1	.	1	1	1	1	1	.	1	.	.	.	.	.	.	.	.
	2828800	$\langle\!\langle 8,6,3,-\rangle\!\rangle$	1	1	1	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.
	2872320	$\langle\!\langle 7,6,3,1\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	2489344	$\langle\!\langle 7,5,3,2\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	282880	$\langle\!\langle 6,5,3,2,1,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\star$	282880	$\langle\!\langle 6,5,3,2,1,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 18.12: 7-modular decomposition matrix of block $\mathfrak{b}_{40}$ of $\tilde{A}_{17}$
(defect 2, bar core $(2, 1)$, weight 2, content $(6, 5, 4, 2)$)

			$[\![7,4,3,2,1]\!]$ 339456	$[\![7^2,2,1]\!]$ 171648	$[\![8,4,3,2]\!]$ 1357824	$[\![8,6,2,1]\!]$ 731648	$[\![8,7,2]\!]$ 1920	$[\![9,4,3,1]\!]$ 1514496	$[\![9,5,2,1]\!]$ 16896	$[\![9,7,1]\!]$ 11392	$[\![10,4,2,1]\!]$ 505344
$\star$	1920	$\langle\!\langle 16, 1\rangle\!\rangle$	.	.	.	.	1	.	.	.	.
$\star$	13312	$\langle\!\langle 15, 2\rangle\!\rangle$	.	.	.	.	1	.	.	1	.
$\star$	28288	$\langle\!\langle 14, 2, 1, +\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
	28288	$\langle\!\langle 14, 2, 1, -\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
$\star$	522240	$\langle\!\langle 11, 3, 2, 1\rangle\!\rangle$	.	.	.	.	.	.	1	.	1
$\star$	2036736	$\langle\!\langle 10, 4, 2, 1\rangle\!\rangle$	.	.	.	.	.	1	1	.	1
$\star$	183040	$\langle\!\langle 9, 8\rangle\!\rangle$	.	1	.	.	.	.	.	1	.
$\star$	933504	$\langle\!\langle 9, 7, 1, +\rangle\!\rangle$	.	1	.	1	1	.	1	1	.
	933504	$\langle\!\langle 9, 7, 1, -\rangle\!\rangle$	.	1	.	1	1	.	1	1	.
	3620864	$\langle\!\langle 9, 5, 2, 1\rangle\!\rangle$	.	.	1	1	.	1	1	.	.
	2872320	$\langle\!\langle 9, 4, 3, 1\rangle\!\rangle$	.	.	1	.	.	1	.	.	.
	1244672	$\langle\!\langle 8, 7, 2, +\rangle\!\rangle$	1	1	.	1	1	.	.	.	.
	1244672	$\langle\!\langle 8, 7, 2, -\rangle\!\rangle$	1	1	.	1	1	.	.	.	.
	3111680	$\langle\!\langle 8, 6, 2, 1\rangle\!\rangle$	2	2	1	1	.	.	.	.	.
$\star$	1697280	$\langle\!\langle 8, 4, 3, 2\rangle\!\rangle$	1	.	1	.	.	.	.	.	.
$\star$	339456	$\langle\!\langle 7, 4, 3, 2, 1, +\rangle\!\rangle$	1	.	.	.	.	.	.	.	.
	339456	$\langle\!\langle 7, 4, 3, 2, 1, -\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 18.13: 7-modular decomposition matrix of block B_{72} of $\tilde{S}_{17}$
(defect 2, bar core $(2,1)$, weight 2, content $(6,5,4,2)$)

			$[\![7,4,3,2,1,+]\!]$ 339456	$[\![7,4,3,2,1,-]\!]$ 339456	$[\![7^2,2,1,+]\!]$ 171648	$[\![7^2,2,1,-]\!]$ 171648	$[\![8,4,3,2,+]\!]$ 1357824	$[\![8,4,3,2,-]\!]$ 1357824	$[\![8,6,2,1,+]\!]$ 731648	$[\![8,6,2,1,-]\!]$ 731648	$[\![8,7,2,+]\!]$ 1920	$[\![8,7,2,-]\!]$ 1920	$[\![9,4,3,1,+]\!]$ 1514496	$[\![9,4,3,1,-]\!]$ 1514496	$[\![9,5,2,1,+]\!]$ 16896	$[\![9,5,2,1,-]\!]$ 16896	$[\![9,7,1,+]\!]$ 11392	$[\![9,7,1,-]\!]$ 11392	$[\![10,4,2,1,+]\!]$ 505344	$[\![10,4,2,1,-]\!]$ 505344
⋆	1920	$\langle\!\langle 16,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
⋆	1920	$\langle\!\langle 16,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
⋆	13312	$\langle\!\langle 15,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.
⋆	13312	$\langle\!\langle 15,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.
⋆	56576	$\langle\!\langle 14,2,1\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	.	.
⋆	522240	$\langle\!\langle 11,3,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1
⋆	522240	$\langle\!\langle 11,3,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.
⋆	2036736	$\langle\!\langle 10,4,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.
⋆	2036736	$\langle\!\langle 10,4,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1
⋆	183040	$\langle\!\langle 9,8,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
⋆	183040	$\langle\!\langle 9,8,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.
⋆	1867008	$\langle\!\langle 9,7,1\rangle\!\rangle$	.	.	1	1	.	.	1	1	1	1	.	.	1	1	1	1	.	.
⋆	3620864	$\langle\!\langle 9,5,2,1,+\rangle\!\rangle$	.	.	.	.	1	.	.	1	.	.	1	.	1	.	.	.	.	.
	3620864	$\langle\!\langle 9,5,2,1,-\rangle\!\rangle$	.	.	.	.	.	1	1	.	.	.	.	1	.	1	.	.	.	.
⋆	2872320	$\langle\!\langle 9,4,3,1,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.
	2872320	$\langle\!\langle 9,4,3,1,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
	2489344	$\langle\!\langle 8,7,2\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
⋆	3111680	$\langle\!\langle 8,6,2,1,+\rangle\!\rangle$	1	1	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.
	3111680	$\langle\!\langle 8,6,2,1,-\rangle\!\rangle$	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.
⋆	1697280	$\langle\!\langle 8,4,3,2,+\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1697280	$\langle\!\langle 8,4,3,2,-\rangle\!\rangle$	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	678912	$\langle\!\langle 7,4,3,2,1\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

19 The 3-modular spin characters of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

The irreducible 3-modular characters of S_{18} and A_{18} have been computed jointly by J. Müller and the author, using essentially the methods presented in this text. A proof of these results will appear elsewhere.

The constants of the spin 3-blocks of $\tilde{S}_{18}$ and $\tilde{A}_{18}$ are given in Table 19.1.

Table 19.1: Spin 3-blocks of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
8	60	11	8	$()$	6	5	60	22
9	6	4	2	$(7,4,1)$	2	6	6	2
10	3	2	1	$(8,5,2)$	1	7	3	1

Irreducible Brauer characters of $\tilde{S}_{18}$ are the basic spin character $\sigma(18) := \sigma\langle\langle 18\rangle\rangle|_{3'}$, and $\sigma(17,1) := \sigma\langle\langle 17,1\rangle\rangle|_{3'} - \sigma(18)$. The 3-modular basic spin characters $\lambda(18)$ and $\lambda(18)^c$ of $\tilde{A}_{18}$ are computed directly as the irreducible constituents of $\sigma(18)\downarrow_{\tilde{A}_{18}}$, see (3.12).

19.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{10} and b_7 are given in Appendix A as Figures A.33 and A.34, respectively.

19.2 The block b_6

The Brauer characters $\underline{\lambda}_i$ and projective characters $\underline{\Lambda}_j$ defined by Table 19.2 satisfy $\langle\underline{\lambda}_i, \underline{\Lambda}_j\rangle = \delta_{ij}$ for all $i, j \in \{1,2\}$. The decomposition matrix of b_6 is Table 19.8.

Table 19.2: Some characters for b_6

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg \underline{\Lambda}_i$
1	$\lambda\langle\langle 13,4,1\rangle\rangle\vert_{3'}$	1492992	$\Lambda[\![5,3^2,2:3,2,+]\!]\uparrow^{\tilde{A}_{18}}\vert_{b_6}/12$	26873856
2	$\lambda\langle\langle 10,7,1\rangle\rangle\vert_{3'}$	4852224	$\Lambda[\![6^2,4,1]\!]\uparrow^{\tilde{A}_{18}}\vert_{b_6}/5$	30233088

19.3 The block B_9

Since $\underline{\lambda}_1(b_6) = \lambda\langle\langle 13,4,1\rangle\rangle|_{3'}$ and $\underline{\lambda}_2(b_6) = \lambda\langle\langle 10,7,1\rangle\rangle|_{3'}$ are the irreducible Brauer characters of b_6 that is covered by B_9, we take $\underline{\sigma}_1 := \sigma\langle\langle 13,4,1,+\rangle\rangle|_{3'}$ with $\underline{\sigma}_1^a = \sigma\langle\langle 13,4,1,-\rangle\rangle|_{3'}$, and $\underline{\sigma}_2 := \sigma\langle\langle 10,7,1,+\rangle\rangle|_{3'}$ with $\underline{\sigma}_2^a = \sigma\langle\langle 10,7,1,-\rangle\rangle|_{3'}$. Then $\underline{\lambda}_i(b_6)\uparrow^{\tilde{S}_{18}} = \underline{\sigma}_i + \underline{\sigma}_i^a$ for $i \in \{1,2\}$. The decomposition matrix of B_9 is Table 19.9.

19.4 The block B_8

We have basic sets $\{\sigma_1,\dots,\sigma_{11}\}$ and $\{\Sigma_1,\dots,\Sigma_{11}\}$ defined by Table 19.3. The matrix of mutual scalar products is Table 19.4.
Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1,2\}$. It is clear that $\underline{\Sigma}_i := \Sigma_i$ is indecomposable for $i \in \{9,10,11\}$. Since $\Sigma_5 = \sigma_5^* + \sigma_6^*$ with $\deg \sigma_5^* < 0$, $\underline{\Sigma}_5 := \Sigma_5$ is indecomposable too. The relation $\sigma_{12} = -3\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_7 + \sigma_9$ and Table 19.4 show that that $\underline{\sigma}_1$ is contained in σ_7 twice and in σ_9 once. We get $\sigma_{7a} := \sigma_7 - 2\underline{\sigma}_1$ and $\sigma_{9a} := \sigma_9 - \underline{\sigma}_1$. Then $\sigma_{13} = -5\underline{\sigma}_1 - \underline{\sigma}_2 + \sigma_3 + 3\sigma_4$ gives $\sigma_{3a} := \sigma_3 - \underline{\sigma}_2$ and $\sigma_{4a} := \sigma_4 - \underline{\sigma}_1$ (the latter because $\underline{\sigma}_1$ can be contained in σ_3 at most two times). We improve Σ_3 by $\Sigma_{3a} := \Sigma_3 - 2\underline{\Sigma}_{11}$ since $\Sigma_{12} = 2\Sigma_3 - 3\underline{\Sigma}_{11}$. Then

$$\sigma_{14} = -\sigma_{3a} - \sigma_{4a} + 2\sigma_5,$$
$$\sigma_{17} = \underline{\sigma}_1 - 2\underline{\sigma}_2 - \sigma_5 + \sigma_6 + 2\sigma_{7a},$$
$$\sigma_{19} = \underline{\sigma}_1 - 2\sigma_{3a} - 2\sigma_{4a} + \sigma_5 - \sigma_6 + 2\sigma_8$$

yield $\sigma_{5a} := \sigma_5 + 2\underline{\sigma}_1 - \sigma_{4a}$, $\sigma_{7b} := \sigma_{7a} - \sigma_2$, and $\sigma_{8a} := \sigma_8 + 2\underline{\sigma}_1 - \sigma_{4a}$. We have $\sigma_{14} = -4\underline{\sigma}_1 - \sigma_{3a} + \sigma_{4a} + 2\sigma_{5a}$, hence we may improve σ_{5a} by $\sigma_{5b} := \sigma_{5a} - \underline{\sigma}_1$ and σ_{4a} by $\sigma_{4b} := \sigma_{4a} + 2\underline{\sigma}_1 - \sigma_{3a}$. Furthermore, $\sigma_{13} = -8\underline{\sigma}_1 + 4\sigma_{3a} + 3\sigma_{4b}$ gives $\sigma_{3b} := \sigma_{3a} - \underline{\sigma}_1$. From $\sigma_{14} = -4\underline{\sigma}_1 + \sigma_{4b} + 2\sigma_{5b}$ we obtain $\sigma_{5c} := \sigma_{5b} - \underline{\sigma}_1$ and $\underline{\sigma}_4 := \sigma_{4b} - 2\underline{\sigma}_1$ which is a Brauer atom, thus irreducible. The mutual scalar products at this point are given in Table 19.5.

Table 19.3: Some characters for B_8

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(18)$	256	$\Sigma[\![3^5, 2]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	6076850688
2	$\sigma(17, 1)$	3840	$\Sigma[\![4, 3^4, 1, -]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	1022886144
3	$\sigma\langle\!\langle 16, 2\rangle\!\rangle\|_{3'}$	30464	$\Sigma[\![5, 3^3, 2 : 2, -]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	3055221504
4	$\sigma\langle\!\langle 15, 2, 1, -\rangle\!\rangle\|_{3'}$	69888	$\Sigma[\![5, 4, 3^2, 2]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	1834140672
5	$\sigma\langle\!\langle 14, 4\rangle\!\rangle\|_{3'}$	435200	$\Sigma[\![6, 4, 3^2, 1, +]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	349360128
6	$\sigma\langle\!\langle 13, 5\rangle\!\rangle\|_{3'}$	974848	$\Sigma[\![6, 5, 3, 2 : 2, -]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	1256352768
7	$\sigma\langle\!\langle 10, 8\rangle\!\rangle\|_{3'}$	1244672	$\Sigma[\![6^2, 4, 1, +]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	393030144
8	$\sigma\langle\!\langle 6, 5, 4, 2, 1, +\rangle\!\rangle\|_{3'}$	1357824	$\Sigma[\![6, 5, 4, 2]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	2106238464
9	$\sigma\langle\!\langle 9, 8, 1, -\rangle\!\rangle\|_{3'}$	2050048	$\Sigma[\![7, 5, 4, 1, +]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	335923200
10	$\sigma\langle\!\langle 8, 4, 3, 2, 1, -\rangle\!\rangle\|_{3'}$	2376192	$\Sigma[\![7, 5, 3, 2]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	342641664
11	$\sigma\langle\!\langle 8, 7, 3, -\rangle\!\rangle\|_{3'}$	8146944	$\Sigma[\![8, 5, 3, 1]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}/3$	399748608
12	$\sigma[\![6, 5, 4, 2]\!]\uparrow^{\tilde{S}_{18}}$	3290112	$\Sigma[\![5, 3^3, 2, 1]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	4911197184
13	$\sigma[\![5, 3^3, 2, 1]\!]\uparrow^{\tilde{S}_{18}}$	235008		
14	$\sigma[\![5, 4, 3^2, 2]\!]\uparrow^{\tilde{S}_{18}}$	774144		
15	$\sigma(18) \otimes \sigma[15, 3]$	165632		
16	$\sigma[\![6, 5, 3, 2, 1]\!]\uparrow^{\tilde{S}_{18}}$	4856832		
17	$\sigma[\![6^2, 4, 1, +]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	3020544		
18	$\sigma[\![7, 5, 3, 2]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	7441920		
19	$\sigma(18) \otimes \sigma[12, 6]$	1984512		
20	$\sigma[\![7, 5, 4, 1, +]\!]\uparrow^{\tilde{S}_{18}}$	8861184		
21	$\sigma\langle\!\langle 7, 6, 5, -\rangle\!\rangle\|_{3'}$	2193408		
22	$\sigma[\![3^3, 2, 1, \pm : 4, 2, \pm]\!]\uparrow^{\tilde{S}_{18}}\|_{B_8}$	4142592		

The relations $\sigma_{16} = -4\underline{\sigma}_1 - 2\sigma_{3b} - \underline{\sigma}_4 + 2\sigma_{8a} + \sigma_{10}$ and $\sigma_{17} = -\sigma_{3b} - \underline{\sigma}_4 - \sigma_{5c} + \sigma_6 + 2\sigma_{7b}$ yield $\sigma_{8b} := \sigma_{8a} - \sigma_{3b}$, $\sigma_{10a} := \sigma_{10} - \underline{\sigma}_4$ and $\sigma_{6a} := \sigma_6 + \underline{\sigma}_1 - \sigma_{3b} - \sigma_{5c}$, respectively. Then $\sigma_{16} = -4\underline{\sigma}_1 + 2\sigma_{8b} + \sigma_{10a}$ and $\sigma_{17} = -\underline{\sigma}_1 - \underline{\sigma}_4 + \sigma_{6a} + 2\sigma_{7b}$ give $\sigma_{10b} := \sigma_{10a} - 2\underline{\sigma}_1$, $\sigma_{8c} := \sigma_{8b} - \underline{\sigma}_1$, and $\sigma_{6b} := \sigma_{6a} - \underline{\sigma}_1 - \underline{\sigma}_4$.

We have $\sigma_{18} = -4\underline{\sigma}_1 + 4\sigma_{6b} - \sigma_{7b} - 2\sigma_{8c} + \sigma_{9a} + 3\sigma_{10b}$. Hence $\underline{\sigma}_1$ is contained one more time in σ_{6b} and $\underline{\sigma}_6 := \sigma_{6b} - \underline{\sigma}_1$ is irreducible because it is a Brauer atom. Then $\sigma_{19} = -\underline{\sigma}_6 + 2\sigma_{8c}$ yields $\sigma_{8d} := \sigma_{8c} - \underline{\sigma}_6$. Moreover,

$$\sigma_{20} = -7\underline{\sigma}_1 - 4\underline{\sigma}_2 - 3\sigma_{3b} - 2\underline{\sigma}_4 + \underline{\sigma}_6 - 2\sigma_{7b} - \sigma_{8c} + 2\sigma_{9a} + \sigma_{11}$$

give $\sigma_{9b} := \sigma_{9a} - 2\underline{\sigma}_2$ and $\sigma_{11a} := \sigma_{11} - 4\underline{\sigma}_1 - 3\sigma_{3b} - 2\underline{\sigma}_4$ instead of σ_{9a} and σ_{11}, respectively. We obtain $\sigma_{21} = 3\underline{\sigma}_2 - \underline{\sigma}_4 + \sigma_{5c} + \underline{\sigma}_6 - \sigma_{8d} + \sigma_{9b}$ which yields $\underline{\sigma}_5 := \sigma_{5b} - \underline{\sigma}_4$, a Brauer

Table 19.4: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}
σ_1	1	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.
σ_3	2	1	1	.	.	.	.	.	.	.	.
σ_4	3	.	1	1	.	.	.	.	.	.	.
σ_5	2	.	1	2	1	.	.	.	.	.	.
σ_6	2	.	1	2	1	1	.	.	.	.	.
σ_7	2	1	.	.	.	.	1	2	.	.	.
σ_8	2	.	2	1	.	2	.	1	.	.	.
σ_9	1	2	.	.	.	.	1	1	1	.	.
σ_{10}	2	.	.	1	.	.	.	1	.	1	.
σ_{11}	10	.	5	2	.	1	.	3	.	.	1

atom. Then $\sigma_{20} = -3\underline{\sigma}_1 - 2\sigma_{7b} - \sigma_{8d} + 2\sigma_{9b} + \sigma_{11a}$ provides $\sigma_{11b} := \sigma_{11a} - 3\underline{\sigma}_1$, and $\sigma_{23} = 12\underline{\sigma}_1 + 4\underline{\sigma}_2 + 8\sigma_{3b} + 11\underline{\sigma}_4 + 4\underline{\sigma}_5 + \underline{\sigma}_6 - \sigma_{8d} + \sigma_{10b}$ gives $\underline{\sigma}_{10} := \sigma_{10b} - \sigma_{8d} + \underline{\sigma}_6$ as well as $\underline{\sigma}_8 := \sigma_{8d} - \underline{\sigma}_6$ that are irreducible. As $\sigma_{18} = 2\underline{\sigma}_2 - \sigma_{7b} + \underline{\sigma}_8 + \sigma_{9b} + 3\underline{\sigma}_{10}$, we may improve σ_{9b} by $\sigma_{9c} := \sigma_{9b} - \sigma_{7b} + 2\underline{\sigma}_8$. Then $\sigma_{20} = -\underline{\sigma}_6 - 5\underline{\sigma}_8 + 2\sigma_{9c} + \sigma_{11b}$ yields $\underline{\sigma}_9 := \sigma_{9c} - \underline{\sigma}_8$ and $\underline{\sigma}_{11} := \sigma_{11b} - \underline{\sigma}_6 - 3\underline{\sigma}_8$, both irreducible, and $\sigma_{21} = 3\underline{\sigma}_2 + \underline{\sigma}_5 + \sigma_{7b} - 3\underline{\sigma}_8 + \sigma_{9c}$ gives $\underline{\sigma}_7 := \sigma_7 - 2\underline{\sigma}_8$ which is irreducible too. We have found all irreducible Brauer characters of B_8 but one which is either σ_{3b} or $\sigma_{3b} - \underline{\sigma}_1$. Let $16a^+$ be the reduced permutation module of S_{18} over $F := GF(3)$ and let $256a^-$ be a basic spin module of $\tilde{S}_{18}$ realized over F. Then $16a^+ \otimes 16a^+$ has a constituent $120a^+$ that affords $\sigma[16, 1^2]$. The direct examination of $64a^- \otimes 120a^+$ with Brauer character $\sigma(18) \otimes \sigma[16, 1^2] = 2\underline{\sigma}_1 + \underline{\sigma}_2 + \sigma_{3b}$ shows that $64a^-$ which affords $\underline{\sigma}_1$ has multiplicity two indeed. Thus $\underline{\sigma}_3 := \sigma_{3b}$ is irreducible. The decomposition matrix of B_8 is given in Tables 19.10 and 19.11.

19.5 The block b_5

Our basic sets consist of the Brauer characters λ_i and projective characters Λ_i defined by Table 19.6. The mutual scalar products are given in Table 19.7.

Set $\underline{\lambda}_i := \lambda_i$ for $i \in \{1, 2\}$. We have $\underline{\sigma}_2(B_8)\downarrow_{\tilde{A}_{18}} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \lambda_3 + \lambda_4$, thus $\underline{\lambda}_3 := \lambda_3 - \underline{\lambda}_2$ and $\underline{\lambda}_4 := \lambda_4 - \underline{\lambda}_1$ are irreducible, see Table 19.7. The relations $\underline{\Sigma}_4(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_7 + \Lambda_8$,

Table 19.5: Mutual scalar products (B_8)

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_{3a}	Σ_4	$\underline{\Sigma}_5$	Σ_6	Σ_7	Σ_8	$\underline{\Sigma}_9$	$\underline{\Sigma}_{10}$	$\underline{\Sigma}_{11}$
$\underline{\sigma}_1$	1	.	.	.	.	.	.	.	.	.	.
$\underline{\sigma}_2$	.	1	.	.	.	.	.	.	.	.	.
σ_{3b}	1	.	1	.	.	.	.	.	.	.	.
$\underline{\sigma}_4$	.	.	.	1	.	.	.	.	.	.	.
σ_{5c}	.	.	.	1	1	.	.	.	.	.	.
σ_6	2	.	1	2	1	1	.	.	.	.	.
σ_{7b}	.	.	.	.	.	.	1	2	.	.	.
σ_{8a}	2	.	1	.	.	2	.	1	.	.	.
σ_{9a}	.	2	.	.	.	.	1	1	1	.	.
σ_{10}	2	.	.	1	.	.	.	1	.	1	.
σ_{11}	10	.	3	2	.	1	.	3	.	.	1

$\underline{\Sigma}_8(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{15} + \Lambda_{16}$, and $\underline{\Sigma}_{10}(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{19} + \Lambda_{20}$ show that $\underline{\Lambda}_i := \Lambda_i$ is indecomposable for $i \in \{7, 8, 15, 16, 19, 20\}$. Moreover, $\underline{\Sigma}_9(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{17} = \lambda^*_{17} + \lambda^*_{18}$ and $\underline{\Sigma}_{11}(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{21} = \lambda^*_{21} + \lambda^*_{22}$ show that the projective atoms $\underline{\Lambda}_i := \lambda^*_i$, $i \in \{17, 18, 21, 22\}$, are projective indecomposable characters.

Since $\underline{\sigma}_4(B_8)\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_3 + \underline{\lambda}_4 - \lambda_5 - \lambda_6 + \lambda_7 = \underline{\Lambda}^*_7 + \underline{\Lambda}^*_8$, clearly $\underline{\lambda}_7 := \underline{\Lambda}^*_7$ and $\underline{\lambda}_8 := \underline{\Lambda}^*_8$ are irreducible Brauer characters. Further irreducible Brauer characters $\underline{\lambda}_9 := \lambda_9 - \underline{\lambda}_2 - \underline{\lambda}_7 - \underline{\lambda}_8$ and $\underline{\lambda}_{10} := \lambda_{10} - \underline{\lambda}_1 - \underline{\lambda}_7 - \underline{\lambda}_8$ can be deduced from $\underline{\sigma}_5(B_8)\downarrow_{\tilde{A}_{18}} = -\underline{\lambda}_1 - \underline{\lambda}_2 - 2\underline{\lambda}_7 - 2\underline{\lambda}_8 + \lambda_9 + \lambda_{10}$. The relation $\underline{\sigma}_6(B_8)\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_3 + \underline{\lambda}_4 - \lambda_5 - \lambda_6 - 2\underline{\lambda}_7 - 2\underline{\lambda}_8 - \underline{\lambda}_9 - \underline{\lambda}_{10} + \lambda_{11} + \lambda_{12}$ gives rise to the improvements $\lambda_{11a} := \lambda_{11} - \underline{\lambda}_7 - \underline{\lambda}_8 - \underline{\lambda}_{10}$ and $\lambda_{12a} := \lambda_{12} - \underline{\lambda}_7 - \underline{\lambda}_8 - \underline{\lambda}_9$.

We have $\Lambda_{10} = \underline{\lambda}^*_9 + \underline{\lambda}^*_{10} + 2\lambda^*_{16} + 4\lambda^*_{19} + 4\lambda^*_{20}$. Since the elements of $\mathrm{IPr}(B_8)$ have pairwise distinct degrees, for every degree there is exactly one pair $\{\Phi, \Phi^c\}$ of conjugate elements from $\mathrm{IPr}(b_6)$. In particular, the set $\{\Phi, \Phi^c\}$ is necessarily invariant under complex conjugation and the lifted Frobenius automorphism (see Section 1). These conditions are sufficient to identify $\underline{\Lambda}_9 := \underline{\lambda}^*_9 + \lambda^*_{16} + 2\lambda^*_{19} + 2\lambda^*_{20}$ and $\underline{\Lambda}_{10} := \underline{\lambda}^*_{10} + \lambda^*_{16} + 2\lambda^*_{19} + 2\lambda^*_{20}$ as the indecomposable constituents of Λ_{10} among all parts of Λ_{10}.

Then $\underline{\Sigma}_3(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_5 + \Lambda_6 - 2\underline{\Lambda}_9 - 2\underline{\Lambda}_{10} - 2\underline{\Lambda}_{19} - 2\underline{\Lambda}_{20} - 2\underline{\Lambda}_{21} - 2\underline{\Lambda}_{22}$ gives $\Lambda_{5a} := \Lambda_5 - 2\underline{\Lambda}_{10}$ and $\Lambda_{6a} := \Lambda_6 - 2\underline{\Lambda}_9$. Furthermore, $\underline{\Sigma}_7(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{13} + \Lambda_{14} - 4\underline{\Lambda}_{21} - 4\underline{\Lambda}_{22}$ yields $\underline{\Lambda}_i := \Lambda_i - 2\underline{\Lambda}_{21} - 2\underline{\Lambda}_{22}$ for $i \in \{13, 14\}$. We get $\underline{\Lambda}_{11} := \Lambda_{11} - \Lambda_{14a} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ and $\underline{\Lambda}_{12} := \Lambda_{12} - \Lambda_{13a} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20}$ from $\underline{\Sigma}_6(B_8)\downarrow_{\tilde{A}_{18}} = \Lambda_{11} + \Lambda_{12} - \Lambda_{13a} - \Lambda_{14a} - 2\underline{\Lambda}_{19} - 2\underline{\Lambda}_{20}$. Then $\underline{\Lambda}_{11}$, $\underline{\Lambda}_{12}$, $\underline{\Lambda}_{13}$, and $\underline{\Lambda}_{14}$ are indecomposable. Consider the decomposition of $\underline{\Sigma}_3(B_8)\downarrow_{\tilde{A}_{18}}$

Table 19.6: Some characters for $\mathfrak{b}_5$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(18)$	128	$\Lambda[\![3^5, 2, -]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	3038425344
2	$\lambda(18)^c$	128	$\Lambda[\![3^5, 2, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	3038425344
3	$\lambda\langle\!\langle 17, 1, +\rangle\!\rangle\|_{3'}$	2048	$\Lambda[\![4, 3^4, 1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	1022886144
4	$\lambda\langle\!\langle 17, 1, -\rangle\!\rangle\|_{3'}$	2048	$\lambda(18) \otimes \Lambda[11, 3^2, 1]\|_{\mathfrak{b}_5}$	8477861760
5	$\lambda\langle\!\langle 16, 2, +\rangle\!\rangle\|_{3'}$	15232	$\lambda(18) \otimes \Lambda[11, 3, 2^2]\|_{\mathfrak{b}_5}$	2219612544
6	$\lambda\langle\!\langle 16, 2, -\rangle\!\rangle\|_{3'}$	15232	$\lambda(18)^c \otimes \Lambda[11, 3, 2^2]\|_{\mathfrak{b}_5}$	2219612544
7	$\lambda\langle\!\langle 15, 3, -\rangle\!\rangle\|_{3'}$	69632	$\Lambda[\![5, 4, 3^2, 2, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	917070336
8	$\sigma[\![5, 3^3, 2, 1, -]\!]\uparrow^{\tilde{A}_{18}}$	117504	$\Lambda[\![5, 4, 3^2, 2, -]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	917070336
9	$\lambda(18) \otimes \lambda[14, 4]$	204416	$\lambda(18)^c \otimes \Lambda[12, 4, 2]\|_{\mathfrak{b}_5}$	2141510400
10	$\lambda(18)^c \otimes \lambda[14, 4]$	204416	$\Lambda[\![6, 4, 3^2, 1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	349360128
11	$\lambda\langle\!\langle 13, 5, -\rangle\!\rangle\|_{3'}$	487424	$\lambda(18) \otimes \Lambda[9, 7, 2]\|_{\mathfrak{b}_5}$	1167333120
12	$\lambda\langle\!\langle 13, 5, +\rangle\!\rangle\|_{3'}$	487424	$\lambda(18)^c \otimes \Lambda[9, 7, 2]\|_{\mathfrak{b}_5}$	1167333120
13	$\lambda(18) \otimes \lambda[11, 7]$	620160	$\lambda(18)^c \otimes \Lambda[8, 6, 4]\|_{\mathfrak{b}_5}$	996012288
14	$\lambda(18)^c \otimes \lambda[11, 7]$	620160	$\lambda(18) \otimes \Lambda[8, 6, 4]\|_{\mathfrak{b}_5}$	996012288
15	$\lambda\langle\!\langle 12, 6, -\rangle\!\rangle\|_{3'}$	792064	$\Lambda[\![6, 5, 4, 2, -]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	1053119232
16	$\sigma[\![3^3, 2, 1, + : 3, 2, 1, -]\!]\uparrow^{\tilde{A}_{18}}$	1188096	$\Lambda[\![6, 5, 4, 2, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	1053119232
17	$\sigma[\![6, 5, 4, 2, +]\!]\uparrow^{\tilde{A}_{18}}$	1645056	$\Lambda[\![7, 5, 4, 1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	335923200
18	$\sigma[\![6, 5, 4, 2, -]\!]\uparrow^{\tilde{A}_{18}}$	1645056	$\Sigma[\![7, 5, 3, 1, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	1274828544
19	$\sigma[\![3^3, 2, 1, + : 4, 2, -]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	2071296	$\Lambda[\![7, 5, 3, 2, -]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	171320832
20	$\sigma[\![3^3, 2, 1, + : 4, 2, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	2071296	$\Lambda[\![7, 5, 3, 2, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	171320832
21	$\lambda\langle\!\langle 8, 7, 2, 1, -\rangle\!\rangle\|_{3'}$	4356352	$\Lambda[\![8, 5, 2, 1, + : 2]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}/3$	399748608
22	$\lambda\langle\!\langle 8, 7, 2, 1, +\rangle\!\rangle\|_{3'}$	4356352	$\Lambda[\![8, 5, 3, 1, +]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_5}$	599622912

Table 19.7: Mutual scalar products (b_5)

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}	Λ_{19}	Λ_{20}	Λ_{21}	Λ_{22}
λ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_3	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_4	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_5	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_6	1	1	1	2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_7	2	2	.	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_8	3	3	.	4	2	2	1	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
λ_9	.	1	.	2	.	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_{10}	1	.	.	.	2	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
λ_{11}	1	1	.	2	2	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.
λ_{12}	1	1	.	2	1	2	1	1	2	1	.	1	.	.	.	.	.	.	.	.	.	.
λ_{13}	.	.	.	.	.	.	.	.	1	.	.	1	1	.	1	1	.	.	.	.	.	.
λ_{14}	.	.	.	.	.	.	.	.	.	.	1	.	.	1	1	1	.	.	.	.	.	.
λ_{15}	2	2	.	2	1	1	1	1	1	.	1	1	.	.	1	1	.	.	.	.	.	.
λ_{16}	7	7	2	9	5	5	4	3	2	2	1	1	.	.	.	1	.	.	.	.	.	.
λ_{17}	.	.	2	2	.	.	.	.	1	.	1	1	1	1	1	2	1	2	.	.	.	.
λ_{18}	.	.	2	2	.	.	.	.	1	.	1	1	1	1	2	1	1	1	.	.	.	.
λ_{19}	10	10	4	17	9	9	6	5	3	4	1	1	.	.	.	.	.	.	1	.	.	.
λ_{20}	10	10	4	17	9	9	5	6	3	4	1	1	.	.	.	.	.	1	.	1	.	.
λ_{21}	5	5	.	14	3	3	1	1	3	.	1	1	2	2	2	1	.	1	.	.	1	2
λ_{22}	5	5	.	14	3	3	1	1	3	.	1	1	2	2	1	2	.	2	.	.	1	1

again. By (1.2), we know that there are 12 elements of $\mathrm{IBr}(b_5)$ that are fixed under complex conjugation. Hence the corresponding elements of $\mathrm{IPr}(b_5)$ are fixed too, see (1.1). Since we already know the 10 non-fixed elements of $\mathrm{IPr}(b_5)$ (namely $\underline{\Lambda}_i$ for $i \in \{7, 8, 15, \dots, 22\}$), it follows that the indecomposable constituents of Λ_{5a} and Λ_{6a} are $\underline{\Lambda}_5 := \Lambda_{5a} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20} - \underline{\Lambda}_{21} - \underline{\Lambda}_{22}$ and $\underline{\Lambda}_6 := \Lambda_{6a} - \underline{\Lambda}_{19} - \underline{\Lambda}_{20} - \underline{\Lambda}_{21} - \underline{\Lambda}_{22}$, respectively. As $\underline{\sigma}_6(B_8)\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_3 + \underline{\lambda}_4 - \lambda_5 - \lambda_6 + \lambda_{11a} + \lambda_{12a}$, the mutual scalar products imply that $\lambda_6 - \underline{\lambda}_3$ is contained in λ_{11a} and $\lambda_5 - \underline{\lambda}_4$ is contained in λ_{12a}. Therefore, we get $\lambda_{5a} := \lambda_5 - \underline{\lambda}_4$, $\lambda_{6a} := \lambda_6 - \underline{\lambda}_3$, $\underline{\lambda}_{11} := \lambda_{11a} - \lambda_{6a}$, and $\underline{\lambda}_{12} := \lambda_{12a} - \lambda_{5a}$. Note that $\underline{\lambda}_{11}$ and $\underline{\lambda}_{12}$ are irreducible. In the same way that we split Λ_{10} into its indecomposable constituents, we can now split Λ_3. This yields $\underline{\Lambda}_3 := \underline{\lambda}_3^* + \lambda_{16}^* + \lambda_{17}^* + \lambda_{18}^* + 2\lambda_{19}^* + 2\lambda_{20}^*$ and $\underline{\Lambda}_4 := \underline{\lambda}_4^* + \lambda_{16}^* + \lambda_{17}^* + \lambda_{18}^* + 2\lambda_{19}^* + 2\lambda_{20}^*$. Then we have $\underline{\sigma}_i(B_8)\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_{2i-1}^* + \underline{\lambda}_{2i}^*$ for all $i \in \{7, \dots, 11\}$. Thus $\underline{\lambda}_i := \underline{\lambda}_i^*$ is an irreducible Brauer character for $i \in \{13, \dots, 22\}$. Hence there is only one pair of irreducibles left, either $\{\lambda_{5a} - \underline{\lambda}_1, \lambda_{6a} - \underline{\lambda}_2\}$ or $\{\lambda_{5a} - \underline{\lambda}_2, \lambda_{6a} - \underline{\lambda}_1\}$. This is solved easily by comparing traces of the elements $t_{(10,8)}$ and $t_{(17,1)^+}$ on the 13184-dimensional irreducible constituent of a tensor product with Brauer character $\lambda(18) \otimes \lambda[16, 1^2] = \underline{\lambda}_1 + \underline{\lambda}_3 + \lambda_{5a}$ (or $\lambda(18)^c \otimes \lambda[16, 1^2] = \underline{\lambda}_2 + \underline{\lambda}_4 + \lambda_{6a}$). Note that all ingredients are readily available from 19.4 since $64a^- \downarrow_{\tilde{A}_{18}}$ and $120a^+ \downarrow_{\tilde{A}_{18}}$ afford $\lambda(18) + \lambda(18)^c$ and $\lambda[16, 1^2]$, respectively. It turns out that $\{\lambda_{5a} - \underline{\lambda}_1, \lambda_{6a} - \underline{\lambda}_2\} \subset \mathrm{IBr}(b_5)$. The decomposition matrix of b_5 is given in Tables 19.12 and 19.13.

Table 19.8: 3-modular decomposition matrix of block b_6 of $\tilde{A}_{18}$
(defect 2, bar core $(7,4,1)$, weight 2, content $(13,5)$)

			$[\![7,4,3^2,1]\!]$ 1492992	$[\![7,6,4,1]\!]$ 4852224
$\star$	1492992	$\langle\!\langle 13,4,1\rangle\!\rangle$	1	.
$\star$	4852224	$\langle\!\langle 10,7,1\rangle\!\rangle$	.	1
	6345216	$\langle\!\langle 10,4,3,1,+\rangle\!\rangle$	1	1
	6345216	$\langle\!\langle 10,4,3,1,-\rangle\!\rangle$	1	1
	6345216	$\langle\!\langle 7,6,4,1,+\rangle\!\rangle$	1	1
	6345216	$\langle\!\langle 7,6,4,1,-\rangle\!\rangle$	1	1

Table 19.9: 3-modular decomposition matrix of block B_9 of $\tilde{S}_{18}$
(defect 2, bar core $(7,4,1)$, weight 2, content $(13,5)$)

			$[\![7,4,3^2,1,+]\!]$ 1492992	$[\![7,4,3^2,1,-]\!]$ 1492992	$[\![7,6,4,1,+]\!]$ 4852224	$[\![7,6,4,1,-]\!]$ 4852224
$\star$	1492992	$\langle\!\langle 13,4,1,+\rangle\!\rangle$	1	.	.	.
$\star$	1492992	$\langle\!\langle 13,4,1,-\rangle\!\rangle$	.	1	.	.
$\star$	4852224	$\langle\!\langle 10,7,1,+\rangle\!\rangle$	.	.	.	1
$\star$	4852224	$\langle\!\langle 10,7,1,-\rangle\!\rangle$	.	.	1	.
	12690432	$\langle\!\langle 10,4,3,1\rangle\!\rangle$	1	1	1	1
	12690432	$\langle\!\langle 7,6,4,1\rangle\!\rangle$	1	1	1	1

Table 19.10: 3-modular decomposition matrix of block B_8 of $\tilde{S}_{18}$ **(upper part)**
(defect 8, bar core (), weight 6, content $(12,6)$)

			$[[3^5,2,1]]$ 256	$[[4,3^4,2]]$ 3840	$[[5,3^4,1]]$ 26368	$[[5,4,3^2,2,1]]$ 43008	$[[6,4,3^2,2]]$ 322560	$[[6,5,3^2,1]]$ 539648	$[[6,5,4,2,1]]$ 182784	$[[6^2,4,2]]$ 874752	$[[7,5,3,2,1]]$ 2149888	$[[7,5,4,2]]$ 984576	$[[8,5,4,1]]$ 6892032
$\star$	256	$\langle\langle 18,+\rangle\rangle$	1	.	.	.	.	.	.	.	.	.	.
	256	$\langle\langle 18,-\rangle\rangle$	1	.	.	.	.	.	.	.	.	.	.
$\star$	4096	$\langle\langle 17,1\rangle\rangle$	1	1	.	.	.	.	.	.	.	.	.
$\star$	30464	$\langle\langle 16,2\rangle\rangle$	1	1	1	.	.	.	.	.	.	.	.
	139264	$\langle\langle 15,3\rangle\rangle$	2	.	2	2	.	.	.	.	.	.	.
$\star$	69888	$\langle\langle 15,2,1,+\rangle\rangle$	2	.	1	1	.	.	.	.	.	.	.
	69888	$\langle\langle 15,2,1,-\rangle\rangle$	2	.	1	1	.	.	.	.	.	.	.
$\star$	435200	$\langle\langle 14,4\rangle\rangle$	1	.	1	2	1	.	.	.	.	.	.
	439296	$\langle\langle 14,3,1,+\rangle\rangle$	2	1	1	2	1	.	.	.	.	.	.
	439296	$\langle\langle 14,3,1,-\rangle\rangle$	2	1	1	2	1	.	.	.	.	.	.
$\star$	974848	$\langle\langle 13,5\rangle\rangle$	1	.	1	2	1	1	.	.	.	.	.
	1005312	$\langle\langle 13,3,2,+\rangle\rangle$	2	1	2	2	1	1	.	.	.	.	.
	1005312	$\langle\langle 13,3,2,-\rangle\rangle$	2	1	2	2	1	1	.	.	.	.	.
	1584128	$\langle\langle 12,6\rangle\rangle$	2	.	2	2	.	2	2	.	.	.	.
	3311616	$\langle\langle 12,5,1,+\rangle\rangle$	2	1	1	2	1	1	1	.	1	.	.
	3311616	$\langle\langle 12,5,1,-\rangle\rangle$	2	1	1	2	1	1	1	.	1	.	.
	4243200	$\langle\langle 12,4,2,+\rangle\rangle$	6	1	4	5	1	2	2	.	1	.	.
	4243200	$\langle\langle 12,4,2,-\rangle\rangle$	6	1	4	5	1	2	2	.	1	.	.
	1723392	$\langle\langle 12,3,2,1\rangle\rangle$	4	.	4	4	.	2	2	.	.	.	.
	1810432	$\langle\langle 11,7\rangle\rangle$	1	1	1	.	.	1	2	1	.	.	.
	4992000	$\langle\langle 11,6,1,+\rangle\rangle$	2	2	1	1	.	1	2	1	1	1	.
	4992000	$\langle\langle 11,6,1,-\rangle\rangle$	2	2	1	1	.	1	2	1	1	1	.
	5431296	$\langle\langle 11,4,3,+\rangle\rangle$	4	3	2	3	1	1	2	1	1	1	.
	5431296	$\langle\langle 11,4,3,-\rangle\rangle$	4	3	2	3	1	1	2	1	1	1	.
	7676928	$\langle\langle 11,4,2,1\rangle\rangle$	6	4	4	5	2	2	4	2	1	1	.
$\star$	1244672	$\langle\langle 10,8\rangle\rangle$	2	1	.	.	.	.	2	1	.	.	.
	13069056	$\langle\langle 10,6,2,+\rangle\rangle$	7	2	3	2	.	2	5	1	1	1	1
	13069056	$\langle\langle 10,6,2,-\rangle\rangle$	7	2	3	2	.	2	5	1	1	1	1
	14074368	$\langle\langle 10,5,3,+\rangle\rangle$	9	3	5	4	1	3	5	1	1	1	1
	14074368	$\langle\langle 10,5,3,-\rangle\rangle$	9	3	5	4	1	3	5	1	1	1	1

Table 19.11: 3-modular decomposition matrix of block B_8 of $\tilde{S}_{18}$ **(lower part)**
(defect 8, bar core (), weight 6, content $(12,6)$)

			$[\![3^5,2,1]\!]$ 256	$[\![4,3^4,2]\!]$ 3840	$[\![5,3^4,1]\!]$ 26368	$[\![5,4,3^2,2,1]\!]$ 43008	$[\![6,4,3^2,2]\!]$ 322560	$[\![6,5,3^2,1]\!]$ 539648	$[\![6,5,4,2,1]\!]$ 182784	$[\![6^2,4,2]\!]$ 874752	$[\![7,5,3,2,1]\!]$ 2149888	$[\![7,5,4,2]\!]$ 984576	$[\![8,5,4,1]\!]$ 6892032
	16293888	$\langle\!\langle 10,5,2,1\rangle\!\rangle$	12	4	6	6	2	4	7	2	1	1	1
⋆	2050048	$\langle\!\langle 9,8,1,+\rangle\!\rangle$	1	2	.	.	.	.	1	1	.	1	.
	2050048	$\langle\!\langle 9,8,1,-\rangle\!\rangle$	1	2	.	.	.	.	1	1	.	1	.
	9900800	$\langle\!\langle 9,7,2,+\rangle\!\rangle$	4	2	2	.	.	1	3	1	.	1	1
	9900800	$\langle\!\langle 9,7,2,-\rangle\!\rangle$	4	2	2	.	.	1	3	1	.	1	1
	17425408	$\langle\!\langle 9,6,3,+\rangle\!\rangle$	12	.	8	4	.	4	6	.	.	.	2
	17425408	$\langle\!\langle 9,6,3,-\rangle\!\rangle$	12	.	8	4	.	4	6	.	.	.	2
	19009536	$\langle\!\langle 9,6,2,1\rangle\!\rangle$	14	.	10	6	.	6	8	.	.	.	2
	9574400	$\langle\!\langle 9,5,4,+\rangle\!\rangle$	8	.	6	4	.	3	4	.	.	.	1
	9574400	$\langle\!\langle 9,5,4,-\rangle\!\rangle$	8	.	6	4	.	3	4	.	.	.	1
	29872128	$\langle\!\langle 9,5,3,1\rangle\!\rangle$	22	6	14	12	2	8	12	2	2	2	2
	9139200	$\langle\!\langle 9,4,3,2\rangle\!\rangle$	4	6	.	2	2	.	2	2	2	2	.
⋆	8146944	$\langle\!\langle 8,7,3,+\rangle\!\rangle$	7	.	3	2	.	1	3	.	.	.	1
	8146944	$\langle\!\langle 8,7,3,-\rangle\!\rangle$	7	.	3	2	.	1	3	.	.	.	1
	8712704	$\langle\!\langle 8,7,2,1\rangle\!\rangle$	6	.	4	2	.	2	3	.	.	.	1
	11202048	$\langle\!\langle 8,6,4,+\rangle\!\rangle$	9	3	5	4	1	2	4	1	.	1	1
	11202048	$\langle\!\langle 8,6,4,-\rangle\!\rangle$	9	3	5	4	1	2	4	1	.	1	1
	28288000	$\langle\!\langle 8,6,3,1\rangle\!\rangle$	20	6	12	10	2	6	10	2	2	2	2
	18765824	$\langle\!\langle 8,5,4,1\rangle\!\rangle$	14	4	8	8	2	4	8	2	2	1	1
⋆	2376192	$\langle\!\langle 8,4,3,2,1,+\rangle\!\rangle$	2	.	.	1	.	.	1	.	1	.	.
	2376192	$\langle\!\langle 8,4,3,2,1,-\rangle\!\rangle$	2	.	.	1	.	.	1	.	1	.	.
	2193408	$\langle\!\langle 7,6,5,+\rangle\!\rangle$	.	3	.	.	1	.	.	1	.	1	.
	2193408	$\langle\!\langle 7,6,5,-\rangle\!\rangle$	.	3	.	.	1	.	.	1	.	1	.
	10723328	$\langle\!\langle 7,6,3,2\rangle\!\rangle$	6	6	2	4	2	2	4	2	2	2	.
	11315200	$\langle\!\langle 7,5,4,2\rangle\!\rangle$	8	4	4	6	2	4	6	2	2	1	.
	3734016	$\langle\!\langle 7,5,3,2,1,+\rangle\!\rangle$	2	.	2	2	.	2	2	.	1	.	.
	3734016	$\langle\!\langle 7,5,3,2,1,-\rangle\!\rangle$	2	.	2	2	.	2	2	.	1	.	.
	1584128	$\langle\!\langle 6,5,4,3\rangle\!\rangle$	2	.	2	2	.	2	2	.	.	.	.
⋆	1357824	$\langle\!\langle 6,5,4,2,1,+\rangle\!\rangle$	.	.	2	1	.	2	1	.	.	.	.
	1357824	$\langle\!\langle 6,5,4,2,1,-\rangle\!\rangle$	.	.	2	1	.	2	1	.	.	.	.

Table 19.12: 3-modular decomposition matrix of block b_5 of $\tilde{\mathrm{A}}_{18}$ **(upper part)**
(defect 8, bar core $()$, weight 6, content $(12,6)$)

			$\llbracket 3^5,2,1,+\rrbracket$	$\llbracket 3^5,2,1,-\rrbracket$	$\llbracket 4,3^4,2,+\rrbracket$	$\llbracket 4,3^4,2,-\rrbracket$	$\llbracket 5,3^4,1,+\rrbracket$	$\llbracket 5,3^4,1,-\rrbracket$	$\llbracket 5,4,3^2,2,1,+\rrbracket$	$\llbracket 5,4,3^2,2,1,-\rrbracket$	$\llbracket 6,4,3^2,2,+\rrbracket$	$\llbracket 6,4,3^2,2,-\rrbracket$	$\llbracket 6,5,3^2,1,+\rrbracket$	$\llbracket 6,5,3^2,1,-\rrbracket$	$\llbracket 6,5,4,2,1,+\rrbracket$	$\llbracket 6,5,4,2,1,-\rrbracket$	$\llbracket 6^2,4,2,+\rrbracket$	$\llbracket 6^2,4,2,-\rrbracket$	$\llbracket 7,5,3,2,1,+\rrbracket$	$\llbracket 7,5,3,2,1,-\rrbracket$	$\llbracket 7,5,4,2,+\rrbracket$	$\llbracket 7,5,4,2,-\rrbracket$	$\llbracket 8,5,4,1,+\rrbracket$	$\llbracket 8,5,4,1,-\rrbracket$
			128	128	1920	1920	13184	13184	21504	21504	161280	161280	269824	269824	91392	91392	437376	437376	1074944	1074944	492288	492288	3446016	3446016
⋆	256	$\langle\!\langle 18\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	2048	$\langle\!\langle 17,1,+\rangle\!\rangle$	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	2048	$\langle\!\langle 17,1,-\rangle\!\rangle$	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	15232	$\langle\!\langle 16,2,+\rangle\!\rangle$	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	15232	$\langle\!\langle 16,2,-\rangle\!\rangle$	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	69632	$\langle\!\langle 15,3,+\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	69632	$\langle\!\langle 15,3,-\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	69888	$\langle\!\langle 15,2,1\rangle\!\rangle$	2	2	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	217600	$\langle\!\langle 14,4,+\rangle\!\rangle$	.	1	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	217600	$\langle\!\langle 14,4,-\rangle\!\rangle$	1	.	.	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
	439296	$\langle\!\langle 14,3,1\rangle\!\rangle$	2	2	1	1	1	1	2	2	1	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	487424	$\langle\!\langle 13,5,+\rangle\!\rangle$	1	.	.	.	1	.	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.
⋆	487424	$\langle\!\langle 13,5,-\rangle\!\rangle$	.	1	.	.	.	1	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.
	1005312	$\langle\!\langle 13,3,2\rangle\!\rangle$	2	2	1	1	2	2	2	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.
	792064	$\langle\!\langle 12,6,+\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	792064	$\langle\!\langle 12,6,-\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	3311616	$\langle\!\langle 12,5,1\rangle\!\rangle$	2	2	1	1	1	1	2	2	1	1	1	1	1	1	.	.	1	1	.	.	.	.
	4243200	$\langle\!\langle 12,4,2\rangle\!\rangle$	6	6	1	1	4	4	5	5	1	1	2	2	2	2	.	.	1	1	.	.	.	.
	861696	$\langle\!\langle 12,3,2,1,+\rangle\!\rangle$	2	2	.	.	2	2	2	2	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	861696	$\langle\!\langle 12,3,2,1,-\rangle\!\rangle$	2	2	.	.	2	2	2	2	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	905216	$\langle\!\langle 11,7,+\rangle\!\rangle$	1	.	.	1	1	.	.	.	.	.	1	.	1	1	1	.	.	.	.	.	.	.
⋆	905216	$\langle\!\langle 11,7,-\rangle\!\rangle$	.	1	1	.	.	1	.	.	.	.	.	1	1	1	.	1	.	.	.	.	.	.
	4992000	$\langle\!\langle 11,6,1\rangle\!\rangle$	2	2	2	2	1	1	1	1	.	.	1	1	2	2	1	1	1	1	1	1	.	.
	5431296	$\langle\!\langle 11,4,3\rangle\!\rangle$	4	4	3	3	2	2	3	3	1	1	1	1	2	2	1	1	1	1	1	1	.	.
⋆	3838464	$\langle\!\langle 11,4,2,1,+\rangle\!\rangle$	3	3	2	2	2	2	2	3	1	1	1	1	2	2	1	1	1	.	.	1	.	.
	3838464	$\langle\!\langle 11,4,2,1,-\rangle\!\rangle$	3	3	2	2	2	2	3	2	1	1	1	1	2	2	1	1	.	1	1	.	.	.
⋆	622336	$\langle\!\langle 10,8,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.
⋆	622336	$\langle\!\langle 10,8,-\rangle\!\rangle$	1	1	.	1	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.
	13069056	$\langle\!\langle 10,6,2\rangle\!\rangle$	7	7	2	2	3	3	2	2	.	.	2	2	5	5	1	1	1	1	1	1	1	1
	14074368	$\langle\!\langle 10,5,3\rangle\!\rangle$	9	9	3	3	5	5	4	4	1	1	3	3	5	5	1	1	1	1	1	1	1	1

Table 19.13: 3-modular decomposition matrix of block b_5 of $\tilde{A}_{18}$ **(lower part)**
(defect 8, bar core $()$, weight 6, content $(12,6)$)

			$[\![3^5,2,1,+]\!]$ 128	$[\![3^5,2,1,-]\!]$ 128	$[\![4,3^4,2,+]\!]$ 1920	$[\![4,3^4,2,-]\!]$ 1920	$[\![5,3^4,1,+]\!]$ 13184	$[\![5,3^4,1,-]\!]$ 13184	$[\![5,4,3^2,2,1,+]\!]$ 21504	$[\![5,4,3^2,2,1,-]\!]$ 21504	$[\![6,4,3^2,2,+]\!]$ 161280	$[\![6,4,3^2,2,-]\!]$ 161280	$[\![6,5,3^2,1,+]\!]$ 269824	$[\![6,5,3^2,1,-]\!]$ 269824	$[\![6,5,4,2,1,+]\!]$ 91392	$[\![6,5,4,2,1,-]\!]$ 91392	$[\![6^2,4,2,+]\!]$ 437376	$[\![6^2,4,2,-]\!]$ 437376	$[\![7,5,3,2,1,+]\!]$ 1074944	$[\![7,5,3,2,1,-]\!]$ 1074944	$[\![7,5,4,2,+]\!]$ 492288	$[\![7,5,4,2,-]\!]$ 492288	$[\![8,5,4,1,+]\!]$ 3446016	$[\![8,5,4,1,-]\!]$ 3446016
⋆	8146944	$\langle\!\langle 10,5,2,1,+\rangle\!\rangle$	6	6	2	2	3	3	3	3	1	1	2	2	3	4	1	1	1	.	.	1	1	.
	8146944	$\langle\!\langle 10,5,2,1,-\rangle\!\rangle$	6	6	2	2	3	3	3	3	1	1	2	2	4	3	1	1	.	1	1	.	.	1
⋆	2050048	$\langle\!\langle 9,8,1\rangle\!\rangle$	1	1	2	2	.	.	.	.	.	.	.	.	1	1	1	1	.	.	1	1	.	.
	9900800	$\langle\!\langle 9,7,2\rangle\!\rangle$	4	4	2	2	2	2	.	.	.	.	1	1	3	3	1	1	.	.	1	1	1	1
	17425408	$\langle\!\langle 9,6,3\rangle\!\rangle$	12	12	.	.	8	8	4	4	.	.	4	4	6	6	.	.	.	.	.	.	2	2
	9504768	$\langle\!\langle 9,6,2,1,+\rangle\!\rangle$	7	7	.	.	5	5	3	3	.	.	3	3	4	4	.	.	.	.	.	.	1	1
	9504768	$\langle\!\langle 9,6,2,1,-\rangle\!\rangle$	7	7	.	.	5	5	3	3	.	.	3	3	4	4	.	.	.	.	.	.	1	1
	9574400	$\langle\!\langle 9,5,4\rangle\!\rangle$	8	8	.	.	6	6	4	4	.	.	3	3	4	4	.	.	.	.	.	.	1	1
	14936064	$\langle\!\langle 9,5,3,1,+\rangle\!\rangle$	11	11	3	3	7	7	6	6	1	1	4	4	6	6	1	1	1	1	1	1	1	1
	14936064	$\langle\!\langle 9,5,3,1,-\rangle\!\rangle$	11	11	3	3	7	7	6	6	1	1	4	4	6	6	1	1	1	1	1	1	1	1
	4569600	$\langle\!\langle 9,4,3,2,+\rangle\!\rangle$	2	2	3	3	.	.	1	1	1	1	.	.	1	1	1	1	1	1	1	1	.	.
	4569600	$\langle\!\langle 9,4,3,2,-\rangle\!\rangle$	2	2	3	3	.	.	1	1	1	1	.	.	1	1	1	1	1	1	1	1	.	.
	8146944	$\langle\!\langle 8,7,3\rangle\!\rangle$	7	7	.	.	3	3	2	2	.	.	1	1	3	3	.	.	.	.	.	.	1	1
⋆	4356352	$\langle\!\langle 8,7,2,1,+\rangle\!\rangle$	3	3	.	.	2	2	1	1	.	.	1	1	1	2	.	.	.	.	.	.	1	.
⋆	4356352	$\langle\!\langle 8,7,2,1,-\rangle\!\rangle$	3	3	.	.	2	2	1	1	.	.	1	1	2	1	.	.	.	.	.	.	.	1
	11202048	$\langle\!\langle 8,6,4\rangle\!\rangle$	9	9	3	3	5	5	4	4	1	1	2	2	4	4	1	1	.	.	1	1	1	1
	14144000	$\langle\!\langle 8,6,3,1,+\rangle\!\rangle$	10	10	3	3	6	6	5	5	1	1	3	3	5	5	1	1	1	1	1	1	1	1
	14144000	$\langle\!\langle 8,6,3,1,-\rangle\!\rangle$	10	10	3	3	6	6	5	5	1	1	3	3	5	5	1	1	1	1	1	1	1	1
⋆	9382912	$\langle\!\langle 8,5,4,1,+\rangle\!\rangle$	7	7	2	2	4	4	4	4	1	1	2	2	4	4	1	1	1	1	.	1	1	.
	9382912	$\langle\!\langle 8,5,4,1,-\rangle\!\rangle$	7	7	2	2	4	4	4	4	1	1	2	2	4	4	1	1	1	1	1	.	.	1
⋆	2376192	$\langle\!\langle 8,4,3,2,1\rangle\!\rangle$	2	2	.	.	.	.	1	1	.	.	.	.	1	1	.	.	1	1	.	.	.	.
	2193408	$\langle\!\langle 7,6,5\rangle\!\rangle$	.	.	3	3	.	.	.	.	1	1	.	.	.	.	1	1	.	.	1	1	.	.
	5361664	$\langle\!\langle 7,6,3,2,+\rangle\!\rangle$	3	3	3	3	1	1	2	2	1	1	1	1	2	2	1	1	1	1	1	1	.	.
	5361664	$\langle\!\langle 7,6,3,2,-\rangle\!\rangle$	3	3	3	3	1	1	2	2	1	1	1	1	2	2	1	1	1	1	1	1	.	.
⋆	5657600	$\langle\!\langle 7,5,4,2,+\rangle\!\rangle$	4	4	2	2	2	2	3	3	1	1	2	2	3	3	1	1	1	1	.	1	.	.
	5657600	$\langle\!\langle 7,5,4,2,-\rangle\!\rangle$	4	4	2	2	2	2	3	3	1	1	2	2	3	3	1	1	1	1	1	.	.	.
	3734016	$\langle\!\langle 7,5,3,2,1\rangle\!\rangle$	2	2	.	.	2	2	2	2	.	.	2	2	2	2	.	.	1	1	.	.	.	.
⋆	792064	$\langle\!\langle 6,5,4,3,+\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	792064	$\langle\!\langle 6,5,4,3,-\rangle\!\rangle$	1	1	.	.	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
	1357824	$\langle\!\langle 6,5,4,2,1\rangle\!\rangle$	.	.	.	.	2	2	1	1	.	.	2	2	1	1	.	.	.	.	.	.	.	.

20 The 5-modular spin characters of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

The irreducible 5-modular characters of S_{18} and A_{18} have been computed using the character-theoretic methods presented in this text. Alternatively, for S_{18} the LLT algorithm can be used (see the GAP3 package Specht [46]) since James' conjecture that in certain cases the decomposition numbers of S_n coincide with the decomposition numbers of a corresponding complex Iwahori–Hecke algebra holds for blocks of weight $\leqslant 4$ ([15]).

Both $\tilde{S}_{18}$ and $\tilde{A}_{18}$ have three spin 5-blocks of defect 0. The block constants of all other spin 5-blocks are collected in Table 20.1.

Table 20.1: Spin 5-blocks of non-zero defect of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
37	27	10	3	$(2,1)$	3	20	30	20
38	30	20	3	(3)	3	21	27	10
39	5	4	1	$(7,4,2)$	1	22	4	2
41	4	2	1	$(9,4)$	1	25	5	4

The 5-modular basic spin characters of $\tilde{S}_{18}$ are $\sigma(18) := \sigma\langle\langle 18,+\rangle\rangle|_{5'}$, $\sigma(18)^a = \sigma\langle\langle 18,-\rangle\rangle|_{5'}$. Another irreducible Brauer character of $\tilde{S}_{18}$ is $\sigma(17,1) := \sigma\langle\langle 17,1\rangle\rangle|_{5'}$. Irreducible Brauer characters of $\tilde{A}_{18}$ are $\lambda(18) := \lambda\langle\langle 18\rangle\rangle|_{5'}$, $\lambda(17,1) := \sigma\langle\langle 17,1,+\rangle\rangle|_{5'}$, $\lambda(17,1)^c = \sigma\langle\langle 17,1,-\rangle\rangle|_{5'}$.

20.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{39}, B_{41}, b_{22}, and and b_{25} are given in Appendix A as Figures A.35, A.36, A.37, and A.38, respectively.

20.2 The block b_{21}

The Brauer characters $\underline{\lambda}_i$ and the projective characters $\underline{\Lambda}_j$ defined by Table 20.2 satisfy $\langle \underline{\lambda}_i, \underline{\Lambda}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, 10\}$. The decomposition matrix of b_{21} is Table 20.9.

Table 20.2: Some characters for b_{21}

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg \underline{\Lambda}_i$
1	$\lambda(18)$	256	$\Lambda[\![5^3, 2, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	320352000
2	$\lambda[\![7, 5, 4, 1, +]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	69376	$\Lambda[\![7, 5, 4, + : 2]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}/2$	137088000
3	$\sigma[\![5^2, 3, 2, 1, +]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	204288	$\Lambda[\![5^2, 4, 2, + : 2]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}/3$	180160000
4	$\lambda[\![7, 6, 4, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	369920	$\Lambda[\![8, 6, 3, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	39712000
5	$\sigma[\![9, 6, 1, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	417792	$\Lambda[\![9, 6, 2, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	56768000
6	$\lambda[\![8, 7, 2, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	635392	$\lambda(18) \otimes \Lambda[11, 7] \lvert_{\mathrm{b}_{21}}/2$	35552000
7	$\lambda[\![6, 5, 4, 2, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	792064	$\Lambda[\![6, 5, 3, 1, + : 2, 1]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}/2$	67008000
8	$\lambda[\![7, 4, 3, 2, 1, +]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	1967616	$\Lambda[\![7, 4, 3, 2, 1, +]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	198016000
9	$\lambda[\![9, 5, 2, 1, +]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	4377600	$\Lambda[\![9, 4, 3, 1]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}/2$	93568000
10	$\lambda[\![7, 5, 3, 2, -]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}$	6349056	$\Lambda[\![7, 5, 2, 1, + : 2, 1]\!]\uparrow^{\tilde{A}_{18}} \lvert_{\mathrm{b}_{21}}/2$	104448000

Table 20.3: Some characters for B_{38}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(18)$	256	$\sigma(18) \otimes \Sigma[4^3, 3^2] \lvert_{\mathrm{B}_{38}}$	320352000
2	$\sigma(18)^a$	256	$\sigma(18) \otimes \Sigma[15, 1^3] \lvert_{\mathrm{B}_{38}}$	320352000
3	$\underline{\lambda}_3(\mathrm{b}_{21})\uparrow^{\tilde{S}_{18}}$	408576	$\Sigma[\![5^2, 4, 3, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	180160000
4	$\sigma[\![5, 4, 1 : 5, 3, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	1736704	$\Sigma[\![5^2, 4, 3, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	180160000
5	$\sigma[\![5^2, 4, 1, - : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	69888	$\Sigma[\![8, 5, 4, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	137088000
6	$\sigma[\![5^2, 4, 1, + : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	69888	$\Sigma[\![6, 5, 3 : 4, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	657984000
7	$\sigma\langle\!\langle 14, 3, 1, -\rangle\!\rangle\lvert_{5'}$	439296	$\Sigma[\![6, 5, 3 : 4, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	657984000
8	$\sigma\langle\!\langle 14, 3, 1, +\rangle\!\rangle\lvert_{5'}$	439296	$\Sigma[\![8, 5, 2, + : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	838912000
9	$\sigma[\![5^2, 3 : 4, 1, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	626944	$\Sigma[\![8, 5, 3 : 2, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	281248000
10	$\sigma[\![5^2, 3 : 4, 1, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	626944	$\Sigma[\![8, 5, 4, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	137088000
11	$\sigma\langle\!\langle 13, 3, 2, -\rangle\!\rangle\lvert_{5'}$	1005312	$\Sigma[\![8, 5, 1 : 4, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	513216000
12	$\sigma\langle\!\langle 13, 3, 2, +\rangle\!\rangle\lvert_{5'}$	1005312	$\Sigma[\![7, 6, 2 : 2, 1, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	79424000
13	$\sigma[\![5^2, 4, 3, -]\!]\uparrow^{\tilde{S}_{18}}$	1405440	$\Sigma[\![9, 5, 3, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	150336000
14	$\sigma[\![5^2, 4, 3, +]\!]\uparrow^{\tilde{S}_{18}}$	1405440	$\Sigma[\![9, 5, 3, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	150336000
15	$\sigma[\![5, 4, 3, 2, 1, + : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	2376192	$\Sigma[\![8, 7, 2]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	71104000
16	$\sigma[\![5, 4, 3, 2, 1, - : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	2376192	$\Sigma[\![8, 4, 3, 2, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	198016000
17	$\sigma\langle\!\langle 11, 4, 3, -\rangle\!\rangle\lvert_{5'}$	5431296	$\Sigma[\![8, 4, 3, 2, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	198016000
18	$\sigma\langle\!\langle 11, 4, 3, +\rangle\!\rangle\lvert_{5'}$	5431296	$\Sigma[\![9, 4, 3, 1, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	187136000
19	$\sigma\langle\!\langle 8, 7, 3, -\rangle\!\rangle\lvert_{5'}$	8146944	$\Sigma[\![7, 5, 3, 2]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	208896000
20	$\sigma\langle\!\langle 8, 7, 3, +\rangle\!\rangle\lvert_{5'}$	8146944	$\Sigma[\![9, 5, 1, + : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	320640000
21	$\sigma[\![5^2, 3, 1 : 4, +]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	54935040		
22	$\sigma[\![5^2, 3, 1 : 4, -]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	54935040		
23	$\sigma[\![5^2, 3, 2, + : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	9543424		
24	$\sigma[\![5^2, 3, 2, - : 3]\!]\uparrow^{\tilde{S}_{18}} \lvert_{\mathrm{B}_{38}}$	9543424		
25	$\sigma\langle\!\langle 8, 6, 4, +\rangle\!\rangle\lvert_{5'}$	11202048		
26	$\sigma\langle\!\langle 8, 6, 4, -\rangle\!\rangle\lvert_{5'}$	11202048		

20.3 The block B_{38}

Basic sets of Brauer characters σ_i and projective characters Σ_i for $i \in \{1, \dots, 20\}$ are defined by Table 20.3. The mutual scalar products are given in Table 20.4. Let $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$.

Since $\underline{\lambda}_3(b_{21})\uparrow^{\tilde{S}_{18}} = \sigma_3 = \Sigma_3^* + \Sigma_4^*$, the Brauer atoms $\underline{\sigma}_3 := \Sigma_3^*$ and $\underline{\sigma}_4 := \Sigma_4^*$ are irreducible. Moreover, Table 20.4 and the relation $\underline{\lambda}_2(b_{21})\uparrow^{\tilde{S}_{18}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 + \sigma_5 + \sigma_5$ imply that $\underline{\sigma}_i := \sigma_i - \underline{\sigma}_1 - \underline{\sigma}_2$ is irreducible for $i \in \{5, 6\}$. We deduce similarly from

$$\begin{aligned}
\underline{\lambda}_4(b_{21})\uparrow^{\tilde{S}_{18}} &= -\underline{\sigma}_5 - \underline{\sigma}_6 + \sigma_7 + \sigma_8,\\
\underline{\lambda}_5(b_{21})\uparrow^{\tilde{S}_{18}} &= -4\underline{\sigma}_1 - 4\underline{\sigma}_2 - 3\underline{\sigma}_5 - 3\underline{\sigma}_6 + \sigma_9 + \sigma_{10},\\
\underline{\lambda}_7(b_{21})\uparrow^{\tilde{S}_{18}} &= -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 3\underline{\sigma}_3 - 3\underline{\sigma}_4 + \sigma_{13} + \sigma_{14}
\end{aligned}$$

that

$$\begin{aligned}
&\underline{\sigma}_7 := \sigma_7 - \underline{\sigma}_5, & &\underline{\sigma}_8 := \sigma_8 - \underline{\sigma}_6,\\
&\underline{\sigma}_9 := \sigma_9 - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 2\underline{\sigma}_5 - \underline{\sigma}_6, & &\underline{\sigma}_{10} := \sigma_{10} - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_5 - 2\underline{\sigma}_6,\\
&\underline{\sigma}_{13} := \sigma_{13} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_3 - 2\underline{\sigma}_4, & &\underline{\sigma}_{14} := \sigma_{14} - \underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_3 - \underline{\sigma}_4
\end{aligned}$$

are irreducible. We obtain further relations $\underline{\lambda}_6(b_{21})\uparrow^{\tilde{S}_{18}} = -\underline{\sigma}_7 - \underline{\sigma}_8 + \sigma_{11} + \sigma_{12}$ and $\underline{\lambda}_8(b_{21})\uparrow^{\tilde{S}_{18}} = -2\underline{\sigma}_3 - 2\underline{\sigma}_4 + \sigma_{15} + \sigma_{16}$. Hence $\underline{\sigma}_{11} := \sigma_{11} - \underline{\sigma}_7$, $\underline{\sigma}_{12} := \sigma_{12} - \underline{\sigma}_8$, and $\underline{\sigma}_i := \sigma_i - \underline{\sigma}_3 - \underline{\sigma}_4$ for $i \in \{15, 16\}$ are irreducible. Then

$$\begin{aligned}
\underline{\lambda}_9(b_{21})\uparrow^{\tilde{S}_{18}} &= -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_9 - \underline{\sigma}_{10} - \underline{\sigma}_{11} - \underline{\sigma}_{12} + \sigma_{17} + \sigma_{18},\\
\underline{\lambda}_{10}(b_{21})\uparrow^{\tilde{S}_{18}} &= -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_7 - \underline{\sigma}_8 - \underline{\sigma}_{11} - \underline{\sigma}_{12} - \underline{\sigma}_{13} - \underline{\sigma}_{14} + \sigma_{19} + \sigma_{20}.
\end{aligned}$$

As the mutual scalar products show that neither $\underline{\sigma}_9$ nor $\underline{\sigma}_{10}$ can be contained in σ_{19} or

Table 20.4: Mutual scalar products (B_{38})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}	Σ_{19}	Σ_{20}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_4	3	3	1	.	2	6	6	7	2	2	4	.	2	1	.	.	.	.	.	6
σ_5	1	1	.	.	1	1	2	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_6	1	1	.	.	.	2	1	1	1	1	1	.	.	.	.	.	.	.	.	.
σ_7	.	.	.	.	1	2	2	1	1	.	2	1	.	.	.	.	.	.	.	.
σ_8	.	.	.	.	.	2	2	1	1	1	2	1	.	.	.	.	.	.	.	.
σ_9	2	2	.	.	2	4	5	3	1	1	2	.	1	.	.	.	.	.	.	2
σ_{10}	2	2	.	.	1	5	4	3	2	2	2	.	.	1	.	.	.	.	.	2
σ_{11}	.	.	.	.	.	1	1	2	1	.	3	1	.	.	1	.	.	.	.	.
σ_{12}	.	.	.	.	.	1	1	2	.	.	3	1	.	.	1	.	.	.	.	.
σ_{13}	1	1	1	2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_{14}	1	1	2	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_{15}	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
σ_{16}	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
σ_{17}	1	1	.	.	.	1	.	5	.	.	3	.	1	1	1	.	.	1	.	3
σ_{18}	1	1	.	.	.	.	1	4	.	.	2	.	1	1	1	.	.	1	.	2
σ_{19}	1	1	.	.	.	2	2	3	1	.	3	1	.	.	1	.	.	.	1	.
σ_{20}	1	1	.	.	.	2	2	2	1	.	3	1	.	.	1	.	.	.	1	.

σ_{20}, the relations

$$\begin{aligned}
\sigma_{21} &= 5\underline{\sigma}_3 + 5\underline{\sigma}_4 + 2\underline{\sigma}_5 + \underline{\sigma}_6 - 2\underline{\sigma}_7 - 3\underline{\sigma}_8 - \underline{\sigma}_{10} - 2\underline{\sigma}_{11} - 4\underline{\sigma}_{12} + \underline{\sigma}_{14}\\
&\quad + 3\underline{\sigma}_{15} + 3\underline{\sigma}_{16} + \sigma_{17} + 3\sigma_{19} + 2\sigma_{20},\\
\sigma_{22} &= 5\underline{\sigma}_3 + 5\underline{\sigma}_4 + \underline{\sigma}_5 + 2\underline{\sigma}_6 - 3\underline{\sigma}_7 - 2\underline{\sigma}_8 - \underline{\sigma}_9 - 4\underline{\sigma}_{11} - 2\underline{\sigma}_{12} + \underline{\sigma}_{13}\\
&\quad + 3\underline{\sigma}_{15} + 3\underline{\sigma}_{16} + \sigma_{18} + 2\sigma_{19} + 3\sigma_{20},\\
\sigma_{23} &= \underline{\sigma}_1 + \underline{\sigma}_2 + 2\underline{\sigma}_3 + 2\underline{\sigma}_4 - \underline{\sigma}_8 - \underline{\sigma}_{12} + \underline{\sigma}_{13} + \underline{\sigma}_{14} + \sigma_{19},\\
\sigma_{24} &= \underline{\sigma}_1 + \underline{\sigma}_2 + 2\underline{\sigma}_3 + 2\underline{\sigma}_4 - \underline{\sigma}_7 - \underline{\sigma}_{11} + \underline{\sigma}_{13} + \underline{\sigma}_{14} + \sigma_{20},\\
\sigma_{25} &= \underline{\sigma}_3 + \underline{\sigma}_4 + \underline{\sigma}_5 + \underline{\sigma}_6 - \underline{\sigma}_{11} - \underline{\sigma}_{14} + \underline{\sigma}_{15} + \underline{\sigma}_{16} + \sigma_{20},\\
\sigma_{26} &= \underline{\sigma}_3 + \underline{\sigma}_4 + \underline{\sigma}_5 + \underline{\sigma}_6 - \underline{\sigma}_{12} - \underline{\sigma}_{13} + \underline{\sigma}_{15} + \underline{\sigma}_{16} + \sigma_{19}
\end{aligned}$$

imply that $\underline{\sigma}_{10}$ and $\underline{\sigma}_{12}$ are contained in σ_{17}, and $\underline{\sigma}_9$ as well as $\underline{\sigma}_{11}$ are contained in σ_{18}. Moreover, it follows that $\underline{\sigma}_8$, $\underline{\sigma}_{12}$ and $\underline{\sigma}_{13}$ are contained in σ_{19}, and $\underline{\sigma}_7$, $\underline{\sigma}_{11}$, and $\underline{\sigma}_{14}$ are contained in σ_{20}. The irreducible Brauer characters that complete $\mathrm{IBr}(B_{38})$ are

$$\begin{aligned}
\underline{\sigma}_{17} &:= \sigma_{17} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_{10} - \underline{\sigma}_{12}, & \underline{\sigma}_{18} &:= \sigma_{18} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_9 - \underline{\sigma}_{11},\\
\underline{\sigma}_{19} &:= \sigma_{19} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_8 - \underline{\sigma}_{12} - \underline{\sigma}_{13}, & \underline{\sigma}_{20} &:= \sigma_{20} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_7 - \underline{\sigma}_{11} - \underline{\sigma}_{14}.
\end{aligned}$$

The decomposition matrix of B_{38} is Table 20.10.

20.4 The block B_{37}

We have basic sets of Brauer characters σ_i and projective characters Σ_i for $i \in \{1, \ldots, 10\}$ defined by Table 20.5. By Table 20.6, $\underline{\sigma}_i := \sigma_i$ is irreducible for $i \in \{1, \ldots, 7, 9\}$.
Since $\Sigma_{11} = \Sigma_5 + \Sigma_8 + \Sigma_9 - \Sigma_{10}$, Σ_{10} (which is indecomposable) is contained in Σ_8. Then

$$\begin{aligned}
\sigma_{11} &= -\underline{\sigma}_2 - \underline{\sigma}_4 + \sigma_8 + 3\underline{\sigma}_9,\\
\sigma_{12} &= 2\underline{\sigma}_1 - \underline{\sigma}_2 - 3\underline{\sigma}_3 + 3\underline{\sigma}_4 + 4\underline{\sigma}_5 - \underline{\sigma}_7 + 3\sigma_8 + \sigma_{10}
\end{aligned}$$

yield $\underline{\sigma}_8 := \sigma_8 - \underline{\sigma}_2 - \underline{\sigma}_3 - \underline{\sigma}_4$ and $\underline{\sigma}_{10} := \sigma_{10} - \underline{\sigma}_7$, both irreducible. The decomposition matrix of B_{37} is Table 20.11.

20.5 The block b_{20}

Consider the Brauer characters λ_i and projective characters Λ_i for $i \in \{1, \ldots, 20\}$ defined by Table 20.7.

Table 20.5: Some characters for B_{37}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(17,1)$	4096	$\Sigma[\![6,5^2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	163808000
2	$\sigma(18)\otimes\sigma[4^3,3,1^3]\,\vert_{B_{37}}$	26368	$\sigma(18)\otimes\Sigma[13,4,1]\,\vert_{B_{37}}$	180064000
3	$\sigma[\![7,6,4]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	43520	$\sigma(18)\otimes\Sigma[13,3,2]\,\vert_{B_{37}}/2$	79424000
4	$\sigma(18)\otimes\sigma[7,6,5]\,\vert_{B_{37}}$	252672	$\Sigma[\![5^2,4,2:2,-]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	219584000
5	$\sigma\langle\!\langle 6,5,4,2,1,-\rangle\!\rangle\vert_{5'}$	1357824	$\Sigma[\![6,5,4,2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	120576000
6	$\sigma(18)\otimes\sigma[12,6]\,\vert_{B_{37}}$	1557760	$\Sigma[\![10,6,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	64896000
7	$\sigma[\![8,7,2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	1679872	$\sigma(18)\otimes\Sigma[11,4,3]\,\vert_{B_{37}}/2$	98624000
8	$\sigma\langle\!\langle 7,6,5,-\rangle\!\rangle\vert_{5'}$	2193408	$\Sigma[\![7,4,3,2,1]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}/2$	249152000
9	$\sigma[\![7,5,3,2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	5623296	$\Sigma[\![7,6,2:2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	51136000
10	$\sigma\langle\!\langle 11,4,2,1\rangle\!\rangle\vert_{5'}$	7676928	$\Sigma[\![9,5,2,1]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}/3$	104448000
11	$\sigma[\![7,5,3,1:2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	18784256	$\Sigma[\![6,5,2,1:4]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	316416000
12	$\sigma[\![5^2,2,1:4,1]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{37}}$	18617856		

Table 20.6: Mutual scalar products (B_{37})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}
σ_1	1	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.
σ_3	.	.	1	.	.	.	.	.	.	.
σ_4	.	.	.	1	.	.	.	.	.	.
σ_5	.	.	.	.	1	.	.	.	.	.
σ_6	.	.	.	.	.	1	.	.	.	.
σ_7	.	.	.	.	.	.	1	.	.	.
σ_8	.	1	1	1	.	.	.	1	.	.
σ_9	.	.	.	.	.	.	.	.	1	.
σ_{10}	.	.	.	.	.	.	1	1	.	1

Table 20.7: Some characters for $\mathfrak{b}_{20}$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(17,1)$	2048	$\Lambda[\![6,5^2,1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	163808000
2	$\lambda(17,1)^c$	2048	$\lambda(18)\otimes\Lambda[9,2^4,1,+]\ \|_{\mathfrak{b}_{20}}$	320416000
3	$\lambda\langle\!\langle 16,2,-\rangle\!\rangle\|_{5'}$	15232	$\Lambda[\![5^3,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	418624000
4	$\lambda\langle\!\langle 16,2,+\rangle\!\rangle\|_{5'}$	15232	$\Lambda[\![7,5,4,1,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	180064000
5	$\sigma[\![7,6,3,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	21760	$\Lambda[\![7,6,4,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	39712000
6	$\sigma[\![7,6,3,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	21760	$\Lambda[\![7,6,4,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	39712000
7	$\lambda(18)\otimes\lambda[10,8]\ \|_{\mathfrak{b}_{20}}$	252672	$\Sigma[\![7,6,3,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	65280000
8	$\lambda\langle\!\langle 11,7,+\rangle\!\rangle\|_{5'}$	905216	$\Lambda[\![6,5,4,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	60288000
9	$\lambda[\![6,5,4,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	678912	$\Lambda[\![6,5,4,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	60288000
10	$\lambda[\![6,5,4,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	678912	$\Lambda[\![9,6,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	117120000
11	$\lambda\langle\!\langle 12,6,-\rangle\!\rangle\|_{5'}$	792064	$\Lambda[\![10,6,1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	64896000
12	$\lambda\langle\!\langle 12,6,+\rangle\!\rangle\|_{5'}$	792064	$\Lambda[\![9,6,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	117120000
13	$\lambda[\![8,7,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	839936	$\Lambda[\![8,7,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	49312000
14	$\lambda[\![8,7,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	839936	$\Lambda[\![8,7,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	49312000
15	$\lambda[\![7,4,3,2,1,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	1870848	$\Lambda[\![7,5,3,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	97920000
16	$\sigma[\![5^2,3,2,1,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	3477504	$\Lambda[\![7,5,3,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	97920000
17	$\lambda[\![7,5,3,2,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	2811648	$\Lambda[\![7,6,3,1]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	51136000
18	$\lambda[\![7,5,3,2,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	2811648	$\Lambda[\![5^2,4,2,1,+]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	219584000
19	$\lambda\langle\!\langle 11,4,2,1,+\rangle\!\rangle\|_{5'}$	3838464	$\Sigma[\![5^3,1,-]\!]\uparrow^{\tilde{A}_{18}}\|_{\mathfrak{b}_{20}}$	794128000
20	$\lambda\langle\!\langle 11,4,2,1,-\rangle\!\rangle\|_{5'}$	3838464	$\lambda_{11}\otimes\Lambda[10,6,2]\ \|_{\mathfrak{b}_{20}}$	84413984000
21	$\lambda\langle\!\langle 11,7,-\rangle\!\rangle\|_{5'}$	905216		

Table 20.8: Mutual scalar products (b_{20})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}	Σ_{19}	Σ_{20}
σ_1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	5
σ_2	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	5
σ_3	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	22
σ_4	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	22
σ_5	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	12
σ_6	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	12
σ_7	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1
σ_8	.	1	1	.	.	.	.	.	.	1	1	1	.	.	.	.	.	1	3	195
σ_9	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	8
σ_{10}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	8
σ_{11}	.	2	1	1	.	.	.	.	.	1	1	1	.	.	.	.	.	.	3	211
σ_{12}	.	2	1	1	.	.	.	.	.	1	1	1	.	.	.	.	.	.	3	212
σ_{13}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	158
σ_{14}	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	158
σ_{15}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	62
σ_{16}	.	.	2	.	.	.	.	2	1	.	.	.	.	.	1	.	.	4	2	57
σ_{17}	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	130
σ_{18}	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	1	1	.	.	130
σ_{19}	.	.	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	1	533
σ_{20}	.	.	1	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	533

We have $\underline{\sigma}_i(B_{37})\downarrow_{\tilde{A}_{18}} = \lambda_{2i-1} + \lambda_{2i}$ for $i \in \{1,3,5,7,9\}$, hence $\underline{\lambda}_j := \lambda_j$ is irreducible for $j \in \{1,2,5,6,9,10,13,14,17,18\}$. It follows from $\underline{\sigma}_{10}(B_{37})\downarrow_{\tilde{A}_{18}} = -\underline{\lambda}_{13} - \underline{\lambda}_{14} + \lambda_{19} + \lambda_{20}$ and Table 20.8 that $\underline{\lambda}_{19} := \lambda_{19} - \underline{\lambda}_{13}$ and $\underline{\lambda}_{20} := \lambda_{20} - \underline{\lambda}_{14}$ are irreducible. Since $\underline{\sigma}_4(B_{37})\downarrow_{\tilde{A}_{18}} = \lambda_7$ and $\underline{\sigma}_8(B_{37})\downarrow_{\tilde{A}_{18}} = \lambda_{15}$, we aim to split λ_7 and λ_{15} into their irreducible, conjugate constituents. Consider $\lambda_{15} = \Lambda^*_{15} + \Lambda^*_{16} + 62\Lambda^*_{20}$ first. Among all possible pairs of "conjugate" parts of λ_{15} there is only one that is invariant under complex conjugation and the Frobenius automorphism, namely $\underline{\lambda}_{15} := \Lambda^*_{15} + 31\Lambda^*_{20}$ and $\underline{\lambda}_{16} := \Lambda^*_{16} + 31\Lambda^*_{20}$.
We use condensation and the trace formula to split λ_7. Let $F := \mathrm{GF}(25)$ and $2048a^-$ be an $F\tilde{A}_{18}$-module that affords $\lambda(17,1)$ or $\lambda(17,1)^c$. We construct $2048a^-$ as a quotient of the tensor product of a basic spin module of $\tilde{A}_{18}$ and the reduced permutation module of S_{18} restricted to A_{18}. In Section 17 we have constructed a module $1597a^+$ of S_{17} over $F_0 := \mathrm{GF}(5)$ that affords $\sigma[9,8]$. Now $1597a^+$ can be extended to a module $1597\mathtt{A}^+$ of S_{18} that affords $\sigma[9,9]$ by means of *amalgamation* as follows. We have $1597\mathtt{A}^+\downarrow_{S_{17}} \cong 1597a^+$ and $1597\mathtt{A}^+\downarrow_{S_{16}\times S_2} \cong 610\mathtt{A}^+ \oplus 987\mathtt{A}^+$ such that $1597\mathtt{A}^+\downarrow_{S_{16}} \cong 610a^+ \oplus 987a^+$ where $610\mathtt{A}^+$ and $987\mathtt{A}^+$ are irreducible modules of $S_{16}\times S_2$ over F_0 that belong to different 5-blocks and restrict irreducibly to S_{16} as $610a^+$ and $987a^+$, respectively. We work with the matrices of $(1,2)$ and $(1,\ldots,17)$ on $1597a^+$, and $(1,2)$, $(1,\ldots,16)$, and $(17,18)$ on $610a^+ \oplus 987a^+$ with respect to suitable symmetry bases of $1597\mathtt{A}^+\downarrow_{S_{17}}$ and $1597\mathtt{A}^+\downarrow_{S_{16}\times S_2}$ such that the restriction to S_{16} coincides, that is, $(1,2)$ and $(1,\ldots,16)$ have the same matrices on both modules. The bases are obtained by means of the standard basis method, see Section 3. Then, without loss, the matrix of $(17,18)$ is $\mathrm{diag}(-I_{610}, I_{987})$. We restrict $1597\mathtt{A}^+$ to A_{18} and condense $V := 2048a^- \otimes 1597\mathtt{A}^+$ over F with respect to the trivial idempotent e of $K := \langle \tilde{s}_1, \tilde{s}_2 \rangle \leqslant \tilde{A}_{18}$ where

$$s_1 := (1,7,6,5,4,3,2)(9,10,12)(11,14,13)(15,18,16),$$
$$s_2 := (2,3,5)(4,7,6)(8,10)(11,14)(12,13)(16,18).$$

Here $K \cong (7:3) \times (7:6) \times 3$ has order 2646 and is contained in the normalizer of $\langle t_{(7,1^11)}, t_{(1^7,7,1^4)} \rangle$ in $\tilde{A}_{18}$. Let $a_1 := t_1$, $a_2 := t_{17} \cdots t_1$, $b_1 := a_1^{a_2} a_1$, and $b_2 := a_1 a_2$. Set $c_1 := b_1 b_2 b_1 b_2^2$, $c_2 := (c_1 b_1 b_2^2 b_1 b_2 c_1 b_2)^3 b_2$, and $c_3 := b_1 c_2 b_1 b_2 c_1$. Our condensation algebra is $C := \langle ec_1e, ec_2e, ec_3e, e\tilde{x}_1e, e\tilde{x}_2e, e\tilde{x}_3e\rangle$ where

$$
\begin{aligned}
x_1 &:= (1,2,8,7,18,6,17,12,10,15,16,14,3,4,11,5)(9,13),\\
x_2 &:= (1,11,14,5,13,18,2)(3,4,16,17)(6,12)(7,15,10,9,8),\\
x_3 &:= (1,8)(2,18,3,6,13,15,11,5,7,14)(9,16)(10,17).
\end{aligned}
$$

Then Ve has two C-composition factors 60a and 60b that correspond to the irreducible constituents of λ_7. Out of the four possible pairs of constituents the traces of $e\tilde{x}_1e$, $e\tilde{x}_2e$, and $e\tilde{x}_3e$ on 60a and 60b identify $\underline{\lambda}_7 := \Lambda^*_{18} + \Lambda^*_{20}$ and $\underline{\lambda}_8 := \Lambda^*_3 + \Lambda^*_{18} + \Lambda^*_{19}$ as elements of $\mathrm{IBr}(b_{20})$.
It remains to find the irreducible constituents of λ_i for $i \in \{3,4,11,12\}$. We have $\underline{\sigma}_2(B_{37})\downarrow_{\tilde{A}_{18}} = -\underline{\lambda}_1 - \underline{\lambda}_2 + \lambda_3 + \lambda_4$ and $\underline{\sigma}_6(B_{37})\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_1 + \underline{\lambda}_2 - \lambda_3 - \lambda_4 + \lambda_{11} + \lambda_{12}$. Since

$$
\begin{aligned}
\lambda_8 &= \underline{\lambda}_2 - \lambda_4 + \underline{\lambda}_7 + \lambda_{11},\\
\lambda_{21} &= \underline{\lambda}_1 - \lambda_3 + \underline{\lambda}_8 + \lambda_{12}
\end{aligned}
$$

we see that λ_{11} contains λ_4 and λ_{12} contains λ_3. In particular, $\underline{\lambda}_2$ is contained in λ_4, and $\underline{\lambda}_1$ is contained in λ_3. Thus $\mathrm{IBr}(b_{20})$ is completed by $\underline{\lambda}_3 := \lambda_3 - \underline{\lambda}_1$, $\underline{\lambda}_4 := \lambda_4 - \underline{\lambda}_2$, $\underline{\lambda}_{11} := \lambda_{11} - \underline{\lambda}_4$, and $\underline{\lambda}_{12} := \lambda_{12} - \underline{\lambda}_3$. The decomposition matrix of b_{20} is Table 20.12.

Table 20.9: 5-modular decomposition matrix of block $\mathfrak{b}_{21}$ of $\tilde{A}_{18}$
(defect 3, bar core (3), weight 3, content $(7,7,4)$)

			$[\![5^2,4,3,1]\!]$ 204288	$[\![5^3,3]\!]$ 256	$[\![6,5,4,3]\!]$ 792064	$[\![8,4,3,2,1]\!]$ 1967616	$[\![8,5,3,2]\!]$ 6349056	$[\![8,5,4,1]\!]$ 69376	$[\![8,6,4]\!]$ 369920	$[\![8,7,3]\!]$ 635392	$[\![9,5,3,1]\!]$ 4377600	$[\![9,6,3]\!]$ 417792
⋆	256	$\langle\!\langle 18\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
⋆	69632	$\langle\!\langle 15,3,+\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	.
	69632	$\langle\!\langle 15,3,-\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	.
⋆	439296	$\langle\!\langle 14,3,1\rangle\!\rangle$	.	.	.	.	.	1	1	.	.	.
⋆	487424	$\langle\!\langle 13,5,+\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	1
	487424	$\langle\!\langle 13,5,-\rangle\!\rangle$	.	1	.	.	.	1	.	.	.	1
	1492992	$\langle\!\langle 13,4,1\rangle\!\rangle$	.	2	.	.	.	1	1	1	.	1
⋆	1005312	$\langle\!\langle 13,3,2\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.
⋆	5431296	$\langle\!\langle 11,4,3\rangle\!\rangle$	.	2	.	.	.	.	.	1	1	1
⋆	622336	$\langle\!\langle 10,8,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	622336	$\langle\!\langle 10,8,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.	1
	14074368	$\langle\!\langle 10,5,3\rangle\!\rangle$	2	4	.	2	.	2	.	.	2	2
	6345216	$\langle\!\langle 10,4,3,1,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.
	6345216	$\langle\!\langle 10,4,3,1,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.
	2050048	$\langle\!\langle 9,8,1\rangle\!\rangle$	1	2	1	.	.	.	.	1	.	1
	17425408	$\langle\!\langle 9,6,3\rangle\!\rangle$	2	4	1	2	1	2	1	1	1	1
	14936064	$\langle\!\langle 9,5,3,1,+\rangle\!\rangle$	1	2	.	2	1	1	.	.	1	.
	14936064	$\langle\!\langle 9,5,3,1,-\rangle\!\rangle$	1	2	.	2	1	1	.	.	1	.
⋆	8146944	$\langle\!\langle 8,7,3\rangle\!\rangle$	.	2	1	.	1	.	1	1	.	.
	11202048	$\langle\!\langle 8,6,4\rangle\!\rangle$	2	2	.	2	1	2	1	.	.	.
	9382912	$\langle\!\langle 8,5,4,1,+\rangle\!\rangle$	1	2	1	1	1	1	.	.	.	.
	9382912	$\langle\!\langle 8,5,4,1,-\rangle\!\rangle$	1	2	1	1	1	1	.	.	.	.
	9517824	$\langle\!\langle 8,5,3,2,+\rangle\!\rangle$	2	2	1	1	1	.	.	.	.	.
	9517824	$\langle\!\langle 8,5,3,2,-\rangle\!\rangle$	2	2	1	1	1	.	.	.	.	.
⋆	2376192	$\langle\!\langle 8,4,3,2,1\rangle\!\rangle$	2	.	.	1	.	.	.	.	.	.
⋆	792064	$\langle\!\langle 6,5,4,3,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
	792064	$\langle\!\langle 6,5,4,3,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.

Table 20.10: 5-modular decomposition matrix of block B_{38} of $\tilde{S}_{18}$
(defect 3, bar core (3), weight 3, content (7, 7, 4))

			$[\![5^2,4,3,1,+]\!]$ 204288	$[\![5^2,4,3,1,-]\!]$ 204288	$[\![5^3,3,+]\!]$ 256	$[\![5^3,3,-]\!]$ 256	$[\![6,5,4,3,+]\!]$ 792064	$[\![6,5,4,3,-]\!]$ 792064	$[\![8,4,3,2,1,+]\!]$ 1967616	$[\![8,4,3,2,1,-]\!]$ 1967616	$[\![8,5,3,2,+]\!]$ 6349056	$[\![8,5,3,2,-]\!]$ 6349056	$[\![8,5,4,1,+]\!]$ 69376	$[\![8,5,4,1,-]\!]$ 69376	$[\![8,6,4,+]\!]$ 369920	$[\![8,6,4,-]\!]$ 369920	$[\![8,7,3,+]\!]$ 635392	$[\![8,7,3,-]\!]$ 635392	$[\![9,5,3,1,+]\!]$ 4377600	$[\![9,5,3,1,-]\!]$ 4377600	$[\![9,6,3,+]\!]$ 417792	$[\![9,6,3,-]\!]$ 417792
⋆	256	$\langle\!\langle 18,+\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	256	$\langle\!\langle 18,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	139264	$\langle\!\langle 15,3\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
⋆	439296	$\langle\!\langle 14,3,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
⋆	439296	$\langle\!\langle 14,3,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.
⋆	974848	$\langle\!\langle 13,5\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	1	1
	1492992	$\langle\!\langle 13,4,1,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	1	1	.	.	1	.	.	1	.
⋆	1492992	$\langle\!\langle 13,4,1,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	1
⋆	1005312	$\langle\!\langle 13,3,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	1005312	$\langle\!\langle 13,3,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
⋆	5431296	$\langle\!\langle 11,4,3,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1
⋆	5431296	$\langle\!\langle 11,4,3,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.
⋆	1244672	$\langle\!\langle 10,8\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
	14074368	$\langle\!\langle 10,5,3,+\rangle\!\rangle$	1	1	2	2	.	.	1	1	.	.	1	1	.	.	.	.	1	1	1	1
	14074368	$\langle\!\langle 10,5,3,-\rangle\!\rangle$	1	1	2	2	.	.	1	1	.	.	1	1	.	.	.	.	1	1	1	1
	12690432	$\langle\!\langle 10,4,3,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.
	2050048	$\langle\!\langle 9,8,1,+\rangle\!\rangle$	1	.	1	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	1
⋆	2050048	$\langle\!\langle 9,8,1,-\rangle\!\rangle$	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.
	17425408	$\langle\!\langle 9,6,3,+\rangle\!\rangle$	1	1	2	2	1	.	1	1	.	1	1	1	1	.	.	1	.	1	1	.
⋆	17425408	$\langle\!\langle 9,6,3,-\rangle\!\rangle$	1	1	2	2	.	1	1	1	1	.	1	1	.	1	1	.	1	.	.	1
	29872128	$\langle\!\langle 9,5,3,1\rangle\!\rangle$	1	1	2	2	.	.	2	2	1	1	1	1	.	.	.	.	1	1	.	.
⋆	8146944	$\langle\!\langle 8,7,3,+\rangle\!\rangle$	.	.	1	1	.	1	.	.	1	.	.	.	.	1	1	.	.	.	.	.
⋆	8146944	$\langle\!\langle 8,7,3,-\rangle\!\rangle$	.	.	1	1	1	.	.	.	.	1	.	.	1	.	.	1	.	.	.	.
	11202048	$\langle\!\langle 8,6,4,+\rangle\!\rangle$	1	1	1	1	.	.	1	1	1	.	1	1	.	1	.	.	.	.	.	.
⋆	11202048	$\langle\!\langle 8,6,4,-\rangle\!\rangle$	1	1	1	1	.	.	1	1	.	1	1	1	1	.	.	.	.	.	.	.
	18765824	$\langle\!\langle 8,5,4,1\rangle\!\rangle$	1	1	2	2	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.
	19035648	$\langle\!\langle 8,5,3,2\rangle\!\rangle$	2	2	2	2	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.
⋆	2376192	$\langle\!\langle 8,4,3,2,1,+\rangle\!\rangle$	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	2376192	$\langle\!\langle 8,4,3,2,1,-\rangle\!\rangle$	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1584128	$\langle\!\langle 6,5,4,3\rangle\!\rangle$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 20.11: 5-modular decomposition matrix of block B_{37} of $\tilde{S}_{18}$ (defect 3, bar core $(2,1)$, weight 3, content $(8,7,3)$)

			$[\![5^3,2,1]\!]$ 252672	$[\![6,5,4,2,1]\!]$ 1357824	$[\![6,5^2,2]\!]$ 4096	$[\![7,5,3,2,1]\!]$ 1870848	$[\![7,5^2,1]\!]$ 26368	$[\![7,6,3,2]\!]$ 5623296	$[\![7,6,4,1]\!]$ 43520	$[\![8,7,2,1]\!]$ 1679872	$[\![9,6,2,1]\!]$ 5997056	$[\![10,6,2]\!]$ 1557760
$\star$	4096	$\langle\!\langle 17,1\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	.
$\star$	30464	$\langle\!\langle 16,2\rangle\!\rangle$	.	.	1	.	1	.	.	.	.	.
$\star$	69888	$\langle\!\langle 15,2,1,+\rangle\!\rangle$	.	.	.	.	1	.	1	.	.	.
	69888	$\langle\!\langle 15,2,1,-\rangle\!\rangle$	.	.	.	.	1	.	1	.	.	.
$\star$	1584128	$\langle\!\langle 12,6\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	1
	3311616	$\langle\!\langle 12,5,1,+\rangle\!\rangle$	.	.	1	.	1	.	1	1	.	1
	3311616	$\langle\!\langle 12,5,1,-\rangle\!\rangle$	.	.	1	.	1	.	1	1	.	1
$\star$	1723392	$\langle\!\langle 12,3,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.
$\star$	1810432	$\langle\!\langle 11,7\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	1
	9517824	$\langle\!\langle 11,5,2,+\rangle\!\rangle$	1	.	1	.	1	.	.	1	1	1
	9517824	$\langle\!\langle 11,5,2,-\rangle\!\rangle$	1	.	1	.	1	.	.	1	1	1
$\star$	7676928	$\langle\!\langle 11,4,2,1\rangle\!\rangle$	.	.	.	.	.	.	.	1	1	.
	4852224	$\langle\!\langle 10,7,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	1	.	1
	4852224	$\langle\!\langle 10,7,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	1	.	1
	13069056	$\langle\!\langle 10,6,2,+\rangle\!\rangle$	2	1	1	1	2	.	1	1	1	1
	13069056	$\langle\!\langle 10,6,2,-\rangle\!\rangle$	2	1	1	1	2	.	1	1	1	1
	16293888	$\langle\!\langle 10,5,2,1\rangle\!\rangle$	2	.	.	2	2	.	.	.	2	.
	19009536	$\langle\!\langle 9,6,2,1\rangle\!\rangle$	2	1	2	2	2	1	1	1	1	.
$\star$	8712704	$\langle\!\langle 8,7,2,1\rangle\!\rangle$	.	1	2	.	.	1	1	1	.	.
$\star$	2193408	$\langle\!\langle 7,6,5,+\rangle\!\rangle$	1	.	.	1	1	.	1	.	.	.
	2193408	$\langle\!\langle 7,6,5,-\rangle\!\rangle$	1	.	.	1	1	.	1	.	.	.
	12690432	$\langle\!\langle 7,6,4,1\rangle\!\rangle$	2	2	2	2	2	1	1	.	.	.
	10723328	$\langle\!\langle 7,6,3,2\rangle\!\rangle$	2	2	2	1	.	1	.	.	.	.
	3734016	$\langle\!\langle 7,5,3,2,1,+\rangle\!\rangle$	2	1	.	1	.	.	.	.	.	.
	3734016	$\langle\!\langle 7,5,3,2,1,-\rangle\!\rangle$	2	1	.	1	.	.	.	.	.	.
$\star$	1357824	$\langle\!\langle 6,5,4,2,1,+\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.
	1357824	$\langle\!\langle 6,5,4,2,1,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.

Table 20.12: 5-modular decomposition matrix of block b_{20} of $\tilde{\mathrm{A}}_{18}$ (defect 3, bar core $(2,1)$, weight 3, content $(8,7,3)$)

			$[\![5^3,2,1,+]\!]$ 126336	$[\![5^3,2,1,-]\!]$ 126336	$[\![6,5,4,2,1,+]\!]$ 678912	$[\![6,5,4,2,1,-]\!]$ 678912	$[\![6,5^2,2,+]\!]$ 2048	$[\![6,5^2,2,-]\!]$ 2048	$[\![7,5,3,2,1,+]\!]$ 935424	$[\![7,5,3,2,1,-]\!]$ 935424	$[\![7,5^2,1,+]\!]$ 13184	$[\![7,5^2,1,-]\!]$ 13184	$[\![7,6,3,2,+]\!]$ 2811648	$[\![7,6,3,2,-]\!]$ 2811648	$[\![7,6,4,1,+]\!]$ 21760	$[\![7,6,4,1,-]\!]$ 21760	$[\![8,7,2,1,+]\!]$ 839936	$[\![8,7,2,1,-]\!]$ 839936	$[\![9,6,2,1,+]\!]$ 2998528	$[\![9,6,2,1,-]\!]$ 2998528	$[\![10,6,2,+]\!]$ 778880	$[\![10,6,2,-]\!]$ 778880
⋆	2048	$\langle\!\langle 17,1,+\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	2048	$\langle\!\langle 17,1,-\rangle\!\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	15232	$\langle\!\langle 16,2,+\rangle\!\rangle$	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
⋆	15232	$\langle\!\langle 16,2,-\rangle\!\rangle$	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
⋆	69888	$\langle\!\langle 15,2,1\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	1	.	.	1	1	.	.	.	.	.	.
⋆	792064	$\langle\!\langle 12,6,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.
⋆	792064	$\langle\!\langle 12,6,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1
	3311616	$\langle\!\langle 12,5,1\rangle\!\rangle$	.	.	.	.	1	1	.	.	1	1	.	.	1	1	1	1	.	.	1	1
⋆	861696	$\langle\!\langle 12,3,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	861696	$\langle\!\langle 12,3,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
⋆	905216	$\langle\!\langle 11,7,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
⋆	905216	$\langle\!\langle 11,7,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
	9517824	$\langle\!\langle 11,5,2\rangle\!\rangle$	1	1	.	.	1	1	.	.	1	1	.	.	.	.	1	1	1	1	1	1
⋆	3838464	$\langle\!\langle 11,4,2,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
⋆	3838464	$\langle\!\langle 11,4,2,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
	4852224	$\langle\!\langle 10,7,1\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.	1	1	.	.	1	1
	13069056	$\langle\!\langle 10,6,2\rangle\!\rangle$	2	2	1	1	1	1	1	1	2	2	.	.	1	1	1	1	1	1	1	1
	8146944	$\langle\!\langle 10,5,2,1,+\rangle\!\rangle$	1	1	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	1	.	.
	8146944	$\langle\!\langle 10,5,2,1,-\rangle\!\rangle$	1	1	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	1	.	.
⋆	9504768	$\langle\!\langle 9,6,2,1,+\rangle\!\rangle$	1	1	.	1	1	1	1	1	1	1	1	.	.	1	1	.	.	1	.	.
	9504768	$\langle\!\langle 9,6,2,1,-\rangle\!\rangle$	1	1	1	.	1	1	1	1	1	1	.	1	1	.	.	1	1	.	.	.
⋆	4356352	$\langle\!\langle 8,7,2,1,+\rangle\!\rangle$	.	.	1	.	1	1	.	.	.	.	.	1	1	.	.	1	.	.	.	.
⋆	4356352	$\langle\!\langle 8,7,2,1,-\rangle\!\rangle$	.	.	.	1	1	1	.	.	.	.	1	.	.	1	1	.	.	.	.	.
⋆	2193408	$\langle\!\langle 7,6,5\rangle\!\rangle$	1	1	.	.	.	.	1	1	1	1	.	.	1	1	.	.	.	.	.	.
	6345216	$\langle\!\langle 7,6,4,1,+\rangle\!\rangle$	1	1	1	1	1	1	1	1	1	1	1	.	.	1	.	.	.	.	.	.
⋆	6345216	$\langle\!\langle 7,6,4,1,-\rangle\!\rangle$	1	1	1	1	1	1	1	1	1	1	.	1	1	.	.	.	.	.	.	.
⋆	5361664	$\langle\!\langle 7,6,3,2,+\rangle\!\rangle$	1	1	1	1	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.
	5361664	$\langle\!\langle 7,6,3,2,-\rangle\!\rangle$	1	1	1	1	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.
	3734016	$\langle\!\langle 7,5,3,2,1\rangle\!\rangle$	2	2	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1357824	$\langle\!\langle 6,5,4,2,1\rangle\!\rangle$	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

21 The 7-modular spin characters of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

The irreducible 7-modular characters of S_{18} and A_{18} have been computed using the character-theoretic methods presented in this text. Alternatively, for S_{18} the LLT algorithm can be used (see the GAP3 package Specht [46]) since James' conjecture that in certain cases the decomposition numbers of S_n coincide with the decomposition numbers of a corresponding complex Iwahori–Hecke algebra holds for blocks of weight $\leqslant 4$ ([15]).

Both $\tilde{S}_{18}$ and $\tilde{A}_{18}$ have six spin 7-blocks of defect 0. The constants of the spin 7-blocks of non-zero defect are given in Table 21.1.

Table 21.1: Spin 7-blocks of non-zero defect of $\tilde{S}_{18}$ and $\tilde{A}_{18}$

i	$k(B_i)$	$l(B_i)$	defect	bar core	weight	i	$k(b_i)$	$l(b_i)$
90	22	18	2	(4)	2	48	17	9
91	5	3	1	$(6,5)$	1	49	7	6
92	17	9	2	$(3,1)$	2	50	22	18
93	7	6	1	$(6,3,2)$	1	51	5	3
94	7	6	1	$(8,2,1)$	1	52	5	3
96	5	3	1	$(9,2)$	1	55	7	6

The 7-modular basic spin characters of $\tilde{S}_{18}$ are $\sigma(18) := \sigma\langle\langle 18,+\rangle\rangle|_{7'}$, $\sigma(18)^a = \sigma\langle\langle 18,-\rangle\rangle|_{7'}$. Moreover, $\sigma(17,1) := \sigma\langle\langle 17,1\rangle\rangle|_{7'}$ is an irreducible Brauer character of $\tilde{S}_{18}$. Consequently, $\lambda(18) := \lambda\langle\langle 18\rangle\rangle|_{7'}$, $\lambda(17,1) := \sigma\langle\langle 17,1,+\rangle\rangle|_{7'}$, and $\lambda(17,1)^c = \sigma\langle\langle 17,1,-\rangle\rangle|_{7'}$ are irreducible Brauer characters of $\tilde{A}_{18}$.

21.1 Spin blocks of defect 1

The Brauer trees of the blocks B_{91}, B_{93}, B_{94}, and B_{96} are given in Appendix A as Figures A.39, A.40, A.41, and A.42, respectively. The Brauer trees of b_{49}, b_{51}, b_{52}, and b_{55} are Figures A.43, A.44, A.45, and A.46, respectively.

21.2 The block b_{48}

Table 21.2 defines Brauer characters $\underline{\lambda}_i$ and projective characters $\underline{\Lambda}_j$ with $\langle \underline{\lambda}_i, \underline{\Lambda}_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, 9\}$. The decomposition matrix of b_{48} is Table 21.8.

Table 21.2: Some characters for b_{48}

i	$\underline{\lambda}_i$	$\deg \underline{\lambda}_i$	$\underline{\Lambda}_i$	$\deg \underline{\Lambda}_i$
1	$\lambda(18)$	256	$\lambda(18) \otimes \Lambda[15, 1^3] \mid_{b_{48}}$	34633984
2	$\lambda[\![10, 6, 1, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	217344	$\Lambda[\![11, 6]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	8730624
3	$\sigma[\![7, 6, 2, 1, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	687616	$\Lambda[\![7, 6, 3, 1, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	41896960
4	$\lambda[\![10, 5, 2, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	1275648	$\Lambda[\![10, 5, 2, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	20246016
5	$\lambda[\![6, 5, 3, 2, 1, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	1357824	$\Lambda[\![6, 4, 3, 2 : 3, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}/2$	47767552
6	$\lambda[\![9, 5, 3, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	2463744	$\Lambda[\![9, 5, 3, +]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	24523520
7	$\sigma[\![8, 6, 2, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	2810880	$\Lambda[\![8, 5, 4, '+']\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	35286272
8	$\lambda[\![10, 4, 3, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	2967552	$\Lambda[\![7, 3, 2, 1, + : 5, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}/2$	19192320
9	$\sigma[\![7, 4, 3, 2, -]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}$	4299776	$\Lambda[\![7, 5, 2, 1, + : 2, 1]\!] \uparrow^{\tilde{A}_{18}} \mid_{b_{48}}/2$	44782080

21.3 The block B_{90}

We have basic sets $\{\sigma_1 \dots, \sigma_{18}\}$ and $\{\Sigma_1 \dots, \Sigma_{18}\}$ defined by Table 21.3. The matrix of mutual scalar products is Table 21.5. Set $\underline{\sigma}_i := \sigma_i$ for $i \in \{1, 2\}$. It follows from $\underline{\lambda}_2(b_{48}) \uparrow^{\tilde{S}_{18}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 + \sigma_3 + \sigma_4$ and Table 21.5 that $\underline{\sigma}_i := \sigma_i - \underline{\sigma}_1 - \underline{\sigma}_2$ is irreducible for $i \in \{3, 4\}$. Similarly, the relations

$$\begin{aligned}
\underline{\lambda}_3(b_{48}) \uparrow^{\tilde{S}_{18}} &= -4\underline{\sigma}_1 - 4\underline{\sigma}_2 - 3\underline{\sigma}_3 - 3\underline{\sigma}_4 + \sigma_5 + \sigma_6, \\
\underline{\lambda}_5(b_{48}) \uparrow^{\tilde{S}_{18}} &= \sigma_7 + \sigma_8, \\
\underline{\lambda}_4(b_{48}) \uparrow^{\tilde{S}_{18}} &= -\underline{\sigma}_3 - \underline{\sigma}_4 + \sigma_9 + \sigma_{10}
\end{aligned}$$

and consideration of the mutual scalar products lead to

$$\begin{aligned}
&\underline{\sigma}_5 := \sigma_5 - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - \underline{\sigma}_3 - 2\underline{\sigma}_4, \qquad && \underline{\sigma}_6 := \sigma_6 - 2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 2\underline{\sigma}_3 - \underline{\sigma}_4, \\
&\underline{\sigma}_7 := \sigma_7, && \underline{\sigma}_8 := \sigma_8, \\
&\underline{\sigma}_9 := \sigma_9 - \underline{\sigma}_4, && \underline{\sigma}_{10} := \sigma_{10} - \underline{\sigma}_3.
\end{aligned}$$

Further irreducible Brauer characters are obtained from $\underline{\lambda}_8(b_{48}) \uparrow^{\tilde{S}_{18}} = -\underline{\sigma}_9 - \underline{\sigma}_{10} + \sigma_{11} + \sigma_{12}$ and $\underline{\lambda}_7(b_{48}) \uparrow^{\tilde{S}_{18}} = -2\underline{\sigma}_1 - 2\underline{\sigma}_2 - 3\underline{\sigma}_5 - 3\underline{\sigma}_6 + \sigma_{13} + \sigma_{14}$. These are $\underline{\sigma}_{11} := \sigma_{11} - \underline{\sigma}_{10}$,

$\underline{\sigma}_{12} := \sigma_{12} - \underline{\sigma}_9$, $\underline{\sigma}_{13} := \sigma_{13} - \underline{\sigma}_1 - \underline{\sigma}_2 - \underline{\sigma}_5 - 2\underline{\sigma}_6$, and $\underline{\sigma}_{14} := \sigma_{14} - \underline{\sigma}_1 - \underline{\sigma}_2 - 2\underline{\sigma}_5 - \underline{\sigma}_6$. Then we can easily split $\underline{\lambda}_6(b_{48})\uparrow^{\tilde{S}_{18}} = \sigma_{15} = \Lambda_6^* + \Lambda_7^*$ and $\underline{\lambda}_9(b_{48})\uparrow^{\tilde{S}_{18}} = \sigma_{17} = \Lambda_{12}^* + 2\Lambda_{16}^*$ (because $\deg \Lambda_{12}^* = 0$) into their irreducible constituents $\underline{\sigma}_{15} := \Lambda_6^*$ and $\underline{\sigma}_{16} := \Lambda_7^*$ respectively $\underline{\sigma}_{17} := \Lambda_{12}^* + \Lambda_{16}^*$ and $\underline{\sigma}_{18} := \Lambda_{16}^*$. The decomposition matrix of B_{90} is Table 21.9.

Table 21.3: Some characters for B_{90}

i	σ_i	$\deg \sigma_i$	Σ_i	$\deg \Sigma_i$
1	$\sigma(18)$	256	$\sigma(18) \otimes \Sigma[6, 2^6] \mid_{B_{90}}$	34633984
2	$\sigma(18)^a$	256	$\sigma(18) \otimes \Sigma[13, 1^5] \mid_{B_{90}}$	34633984
3	$\sigma[\![7, 6, 1 : 4, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	217856	$\Sigma[\![11, 6, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	8730624
4	$\sigma[\![7, 6, 1 : 4, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	217856	$\Sigma[\![11, 5 : 2, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	28976640
5	$\sigma[\![7, 4 : 6, 1, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	1340672	$\Sigma[\![11, 4, - : 3]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	48168960
6	$\sigma[\![7, 4 : 6, 1, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	1340672	$\Sigma[\![9, 4, + : 5]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	130645760
7	$\sigma\langle\!\langle 6, 5, 4, 2, 1, -\rangle\!\rangle\mid_{7'}$	1357824	$\Sigma[\![9, 4, - : 5]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	130645760
8	$\sigma\langle\!\langle 6, 5, 4, 2, 1, +\rangle\!\rangle\mid_{7'}$	1357824	$\Sigma[\![11, 6, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	8730624
9	$\sigma\langle\!\langle 13, 4, 1, -\rangle\!\rangle\mid_{7'}$	1492992	$\Sigma[\![11, 4, + : 3]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	48168960
10	$\sigma\langle\!\langle 13, 4, 1, +\rangle\!\rangle\mid_{7'}$	1492992	$\Sigma[\![7, 6, 4, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	41896960
11	$\sigma\langle\!\langle 12, 4, 2, -\rangle\!\rangle\mid_{7'}$	4243200	$\Sigma[\![7, 6, 4, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	41896960
12	$\sigma\langle\!\langle 12, 4, 2, +\rangle\!\rangle\mid_{7'}$	4243200	$\Sigma[\![7, 5, 4 : 2, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	121965312
13	$\sigma[\![7, 6, 4, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	4874240	$\Sigma[\![6, 5, + : 4, 2, 1]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	47767552
14	$\sigma[\![7, 6, 4, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	4874240	$\Sigma[\![6, 5, 4, 2, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	47767552
15	$\sigma[\![9, 5, 3]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	4927488	$\sigma(18) \otimes \Sigma[15, 3] \mid_{B_{90}}$	40492032
16	$\sigma\langle\!\langle 11, 4, 3, -\rangle\!\rangle\mid_{7'}$	5431296	$\Sigma[\![7, 4, 3, 1 : 2, 1, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}/2$	89564160
17	$\sigma[\![7, 4, 3, 2 : 2]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	8599552	$\Sigma[\![8, 6 : 4, -]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	90454784
18	$\sigma\langle\!\langle 9, 5, 4, -\rangle\!\rangle\mid_{7'}$	9574400	$\Sigma[\![8, 6 : 4, +]\!]\uparrow^{\tilde{S}_{18}} \mid_{B_{90}}$	90454784

21.4 The block B_{92}

We have $\langle \underline{\sigma}_i, \underline{\Sigma}_j \rangle = \delta_{ij}$ for all Brauer characters $\underline{\sigma}_i$ and projective characters $\underline{\Sigma}_j$ defined by Table 21.4 ($i, j \in \{1, \ldots, 9\}$). The decomposition matrix of B_{92} is Table 21.10.

21.5 The block b_{50}

Table 21.6 defines basic sets $\{\lambda_1, \ldots, \lambda_{18}\}$ and $\{\Lambda_1, \ldots, \Lambda_{18}\}$ for b_{50}. The matrix of mutual scalar products is Table 21.7

Recall that $\underline{\lambda}_1 := \lambda_1$ and $\underline{\lambda}_2 := \lambda_2$ are irreducible. We have $\underline{\sigma}_i(B_{92})\downarrow_{\tilde{A}_{18}} = \lambda_{2i-1} + \lambda_{2i}$

Table 21.4: Some characters for B_{92}

i	$\underline{\sigma}_i$	$\deg \underline{\sigma}_i$	$\underline{\Sigma}_i$	$\deg \underline{\Sigma}_i$
1	$\sigma(17,1)$	4096	$\Sigma[\![8,7,2,-]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	26141696
2	$\sigma(18)\otimes\sigma[14,4]\,\vert_{B_{92}}$	135168	$\Sigma[\![10,5,-:3]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	11966976
3	$\sigma[\![10,5,2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	304128	$\Sigma[\![10,5:2,1]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	28600320
4	$\sigma(18)\otimes\sigma[4^3,3^2]\,\vert_{B_{92}}$	1109504	$\Sigma[\![7^2,2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	83819008
5	$\sigma[\![10,4,1:2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	1419264	$\Sigma[\![10,4,2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	30707712
6	$\sigma[\![8,6,2,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	3299328	$\Sigma[\![8,5,1:4]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}/2$	100452352
7	$\sigma[\![7,4,3,2,1,-]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	3734016	$\Sigma[\![6,5,3,2:2,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	99373568
8	$\sigma[\![9,5,3]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	11271168	$\Sigma[\![9,4,3,1,+]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}/2$	58856448
9	$\sigma[\![7,5,3,2]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}$	15301632	$\Sigma[\![6,5,3:4]\!]\uparrow^{\tilde{S}_{18}}\,\vert_{B_{92}}/2$	77195776

for $i \in \{3,5,6\}$, $\underline{\sigma}_8(B_{92})\downarrow_{\tilde{A}_{18}} = \lambda_{13}+\lambda_{14}$, and $\underline{\sigma}_9(B_{92})\downarrow_{\tilde{A}_{18}} = \lambda_{15}+\lambda_{16}$. Hence $\underline{\lambda}_i := \lambda_i$ is irreducible for $i \in \{5,6,9,\dots,16\}$.

The irreducible, conjugate constituents of $\underline{\sigma}_7(B_{92})\downarrow_{\tilde{A}_{18}} = \lambda_{17} = \Lambda^*_{16}+\Lambda^*_{17}+30\Lambda^*_{18}$ are $\underline{\lambda}_{17} := \Lambda^*_{16}+15\Lambda^*_{18}$ and $\underline{\lambda}_{18} := \Lambda^*_{17}+15\Lambda^*_{18}$ since $\{\underline{\lambda}_{17},\underline{\lambda}_{18}\}$ is the only suitable pair of parts of λ_{17} that is invariant under complex conjugation and the Frobenius automorphism.

We are left with $\underline{\sigma}_2(B_{92})\downarrow_{\tilde{A}_{18}} = -\underline{\lambda}_1-\underline{\lambda}_2+\lambda_3+\lambda_4$ and $\underline{\sigma}_3(B_{92})\downarrow_{\tilde{A}_{18}} = \underline{\lambda}_1+\underline{\lambda}_2-\lambda_3-\lambda_4+\lambda_7+\lambda_8$. The latter relation implies that λ_4 is contained either in λ_7 or in λ_8, and the former relation implies that either $\underline{\lambda}_1$ or $\underline{\lambda}_2$ is contained in λ_4. Because of the first column of Table 21.7 only $\underline{\lambda}_1$ can be contained in λ_4. Thus $\underline{\lambda}_4 := \lambda_4-\underline{\lambda}_1$ and $\underline{\lambda}_3 := \lambda_3-\underline{\lambda}_2$ are irreducible. It remains to decide whether $\varphi := \lambda_7-\underline{\lambda}_3$ and $\varphi^c = \lambda_8-\underline{\lambda}_4$, or $\psi := \lambda_7-\underline{\lambda}_4$ and $\psi^c = \lambda_8-\underline{\lambda}_3$ complete $\mathrm{IBr}(b_{50})$. We employ the trace formula here. Let $F_0 := \mathrm{GF}(7)$. As in 20.5, we use amalgamation to construct an F_0S_{18}-module $4334\mathtt{A}^+$ that affords $\sigma[9^2]$. This is possible because $4334\mathtt{A}^+\downarrow_{S_{17}} \cong 4334\mathtt{a}^+$ (which has been constructed in 18.5) and $4334\mathtt{A}^+\downarrow_{S_{16}\times S_2} \cong 2993\mathtt{A}^+ \oplus 1341\mathtt{A}^+$ such that $4334\mathtt{A}^+\downarrow_{S_{16}} \cong 2993\mathtt{a}^+ \oplus 1341\mathtt{a}^+$ with $2993\mathtt{A}^+$ and $1341\mathtt{A}^+$ are irreducible modules of $S_{16}\times S_2$ over F_0 that belong to different 7-blocks and restrict irreducibly to S_{16} as $2993\mathtt{a}^+$ and $1341\mathtt{a}^+$, respectively. Let $256\mathtt{a}^-$ be a basic spin represensation of $\tilde{A}_{18}$ over $F := \mathrm{GF}(49)$ and restrict $4334\mathtt{A}^+$ to A_{18}. We condense $V := 256\mathtt{a}^- \otimes 4334\mathtt{A}^+$ over F with respect to the trivial idempotent e of the subgroup $K := \langle zt_{(5,1^{13})}, zt_{(1^5,5,1^8)}, zt_{(1^{10},5,1^3)}\rangle \cong 5^3$ of $\tilde{A}_{18}$. Then Ve has dimension 10752 and exactly two composition factors corresponding to $\{\varphi,\varphi^c\}$ or $\{\psi,\psi^c\}$ for the Hecke algebra $eF\tilde{A}_{18}e$. In fact, we find these composition factors for the following condensation

Table 21.5: Mutual scalar products (B_{90})

$\langle\cdot,\cdot\rangle$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}	Σ_{17}	Σ_{18}
σ_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
σ_3	1	1	1	1	1	1	2	.	.	.	.	.	.	.	.	.	2	2
σ_4	1	1	.	.	.	2	1	1	1	.	.	.	.	.	.	.	2	2
σ_5	2	2	1	1	1	5	4	2	2	1	.	.	.	.	.	.	6	6
σ_6	2	2	2	2	2	4	5	1	1	.	1	1	.	.	.	.	6	6
σ_7	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
σ_8	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
σ_9	.	.	.	1	1	3	3	1	1	.	.	.	.	.	1	.	2	3
σ_{10}	.	.	1	1	1	3	3	.	1	.	.	.	.	.	1	.	3	2
σ_{11}	.	.	.	.	1	2	2	.	1	.	.	.	.	.	1	.	1	.
σ_{12}	.	.	.	1	1	2	2	.	1	.	.	.	.	.	1	.	.	1
σ_{13}	1	1	.	.	.	.	.	.	.	1	2	2	.	.	.	.	.	1
σ_{14}	1	1	.	.	.	.	.	.	.	2	1	2	.	.	.	.	1	.
σ_{15}	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.
σ_{16}	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.	.	.	.
σ_{17}	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	2	.	.
σ_{18}	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	.	1

Table 21.6: Some characters for $\mathfrak{b}_{50}$

i	λ_i	$\deg \lambda_i$	Λ_i	$\deg \Lambda_i$
1	$\lambda(17,1)$	2048	$\lambda(18) \otimes \Lambda[9,2^2,1^5,+] \mid_{\mathfrak{b}_{50}}$	77572096
2	$\lambda(17,1)^c$	2048	$\lambda(18) \otimes \Lambda[16,2] \mid_{\mathfrak{b}_{50}}$	26141696
3	$\lambda\langle\langle 15,3,+\rangle\rangle\mid_{7'}$	69632	$\Lambda[\![7^2,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	60963840
4	$\lambda\langle\langle 15,3,-\rangle\rangle\mid_{7'}$	69632	$\Lambda[\![7^2,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	60963840
5	$\lambda[\![10,5,2,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	152064	$\lambda(17,1) \otimes \Lambda[13,4,1] \mid_{\mathfrak{b}_{50}}/4$	11966976
6	$\lambda[\![10,5,2,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	152064	$\Sigma[\![10,5,1,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	14300160
7	$\lambda\langle\langle 10,8,-\rangle\rangle\mid_{7'}$	622336	$\Lambda[\![10,5,2,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	14300160
8	$\lambda\langle\langle 10,8,+\rangle\rangle\mid_{7'}$	622336	$\Lambda[\![10,4,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	15353856
9	$\lambda[\![10,4,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	709632	$\Lambda[\![10,4,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	15353856
10	$\lambda[\![10,4,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	709632	$\Lambda[\![8,6,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	50226176
11	$\lambda[\![8,6,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	1649664	$\Lambda[\![8,6,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	50226176
12	$\lambda[\![8,6,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	1649664	$\Sigma[\![8,5,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	88824064
13	$\lambda[\![9,5,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	5635584	$\Sigma[\![8,5,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	88824064
14	$\lambda[\![9,5,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	5635584	$\Lambda[\![9,5,3,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	29428224
15	$\lambda[\![7,5,3,2,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	7650816	$\Lambda[\![9,5,3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	29428224
16	$\lambda[\![7,5,3,2,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	7650816	$\Lambda[\![7,5,3,2,-]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	88284672
17	$\lambda[\![6,5,3,2,1,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	3734016	$\Lambda[\![7,5,3,2,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	88284672
18	$\lambda[\![7,5,2,1,+:3,+]\!]\uparrow^{\tilde{A}_{18}} \mid_{\mathfrak{b}_{50}}$	9187072	$\lambda_7 \otimes \Lambda[13,4,1] \mid_{\mathfrak{b}_{50}}$	27284818176

Table 21.7: Mutual scalar products (b_{50})

$\langle\cdot,\cdot\rangle$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}
λ_1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4
λ_2	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	4
λ_3	3	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	26
λ_4	4	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	26
λ_5	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	17
λ_6	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	17
λ_7	3	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	53
λ_8	3	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	54
λ_9	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	25
λ_{10}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	25
λ_{11}	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	59
λ_{12}	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	59
λ_{13}	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	113
λ_{14}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	113
λ_{15}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	116
λ_{16}	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	116
λ_{17}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	30
λ_{18}	1	.	2	2	.	.	1	.	.	2	1	2	1	.	.	1	.	335

algebra C. Let $a_1 := t_1$, $a_2 := t_{17} \cdots t_1$, $b_1 := a_1^{a_2} a_1$, and $b_2 := a_1 a_2$. Set $c_1 := (b_1 b_2)^3 b_2$, $c_2 := (b_1 b_2)^4 b_2$, $c_3 := (c_1 c_2)^3 c_2 c_1$, $c_4 := c_3 c_2 c_1^2 c_2^2 c_1$, and $c_5 := c_4 c_1 c_2^2 c_1$. Then define $d_1 := c_5 c_2 c_5 c_1 c_2^2 c_1$, $d_2 := d_1^3$ and $d_3 := c_5 c_2 c_5 d_1^2 c_5$. Finally, let

$$x := (1, 7, 6, 18, 10, 13, 5, 17, 4)(2, 14)(8, 12)(9, 15)(11, 16).$$

Then $C := \langle e d_1 e, e d_2 e, e d_3 e, e \tilde{x} e \rangle$. The trace formula for $e\tilde{x}e$ determines ψ and ψ^c as elements of $\mathrm{IBr}(b_{50})$. The decomposition matrix of b_{50} is Table 21.11.

Table 21.8: 7-modular decomposition matrix of block b_{48} of $\tilde{A}_{18}$ (defect 2, bar core (4), weight 2, content (5, 5, 5, 3))

			$[\![6,5,4,2,1]\!]$ 1357824	$[\![7,5,4,2]\!]$ 4299776	$[\![7,6,4,1]\!]$ 687616	$[\![7^2,4]\!]$ 256	$[\![8,6,4]\!]$ 2810880	$[\![9,5,4]\!]$ 2463744	$[\![11,4,3]\!]$ 2967552	$[\![11,5,2]\!]$ 1275648	$[\![11,6,1]\!]$ 217344
⋆	256	$\langle\!\langle 18\rangle\!\rangle$	.	.	.	1	.	.	.	.	.
⋆	217600	$\langle\!\langle 14,4,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	1
	217600	$\langle\!\langle 14,4,-\rangle\!\rangle$	.	.	.	1	.	.	.	.	1
⋆	1492992	$\langle\!\langle 13,4,1\rangle\!\rangle$	.	.	.	.	.	.	.	1	1
⋆	4243200	$\langle\!\langle 12,4,2\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
⋆	905216	$\langle\!\langle 11,7,+\rangle\!\rangle$	.	.	1	1	.	.	.	.	1
	905216	$\langle\!\langle 11,7,-\rangle\!\rangle$	.	.	1	1	.	.	.	.	1
⋆	4992000	$\langle\!\langle 11,6,1\rangle\!\rangle$	.	.	1	2	1	.	.	1	1
	9517824	$\langle\!\langle 11,5,2\rangle\!\rangle$	.	.	.	.	1	1	1	1	.
⋆	5431296	$\langle\!\langle 11,4,3\rangle\!\rangle$	.	.	.	.	.	1	1	.	.
	9574400	$\langle\!\langle 9,5,4\rangle\!\rangle$	.	1	.	.	1	1	.	.	.
	11202048	$\langle\!\langle 8,6,4\rangle\!\rangle$	2	1	2	2	1	.	.	.	.
	6345216	$\langle\!\langle 7,6,4,1,+\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
	6345216	$\langle\!\langle 7,6,4,1,-\rangle\!\rangle$	1	1	1	.	.	.	.	.	.
⋆	5657600	$\langle\!\langle 7,5,4,2,+\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
	5657600	$\langle\!\langle 7,5,4,2,-\rangle\!\rangle$	1	1	.	.	.	.	.	.	.
⋆	1357824	$\langle\!\langle 6,5,4,2,1\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 21.9: 7-modular decomposition matrix of block B_{90} of $\tilde{S}_{18}$
(defect 2, bar core (4), weight 2, content $(5,5,5,3)$)

			$[\![6,5,4,2,1,+]\!]$ 1357824	$[\![6,5,4,2,1,-]\!]$ 1357824	$[\![7,5,4,2,+]\!]$ 4299776	$[\![7,5,4,2,-]\!]$ 4299776	$[\![7,6,4,1,+]\!]$ 687616	$[\![7,6,4,1,-]\!]$ 687616	$[\![7^2,4,+]\!]$ 256	$[\![7^2,4,-]\!]$ 256	$[\![8,6,4,+]\!]$ 2810880	$[\![8,6,4,-]\!]$ 2810880	$[\![9,5,4,+]\!]$ 2463744	$[\![9,5,4,-]\!]$ 2463744	$[\![11,4,3,+]\!]$ 2967552	$[\![11,4,3,-]\!]$ 2967552	$[\![11,5,2,+]\!]$ 1275648	$[\![11,5,2,-]\!]$ 1275648	$[\![11,6,1,+]\!]$ 217344	$[\![11,6,1,-]\!]$ 217344
⋆	256	$\langle\!\langle 18,+\rangle\!\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
⋆	256	$\langle\!\langle 18,-\rangle\!\rangle$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
⋆	435200	$\langle\!\langle 14,4\rangle\!\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	1
⋆	1492992	$\langle\!\langle 13,4,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
⋆	1492992	$\langle\!\langle 13,4,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
⋆	4243200	$\langle\!\langle 12,4,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
⋆	4243200	$\langle\!\langle 12,4,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
⋆	1810432	$\langle\!\langle 11,7\rangle\!\rangle$	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	1	1
⋆	4992000	$\langle\!\langle 11,6,1,+\rangle\!\rangle$	.	.	.	.	1	.	1	1	.	1	.	.	.	.	1	.	.	1
⋆	4992000	$\langle\!\langle 11,6,1,-\rangle\!\rangle$	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	1	.
	9517824	$\langle\!\langle 11,5,2,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	1	.	.
⋆	9517824	$\langle\!\langle 11,5,2,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	1	1	.	.	1	1	.	.	.
⋆	5431296	$\langle\!\langle 11,4,3,+\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	5431296	$\langle\!\langle 11,4,3,-\rangle\!\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
⋆	9574400	$\langle\!\langle 9,5,4,+\rangle\!\rangle$	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.	.	.	.
⋆	9574400	$\langle\!\langle 9,5,4,-\rangle\!\rangle$	.	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
	11202048	$\langle\!\langle 8,6,4,+\rangle\!\rangle$	1	1	.	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.
⋆	11202048	$\langle\!\langle 8,6,4,-\rangle\!\rangle$	1	1	1	.	1	1	1	1	.	1	.	.	.	.	.	.	.	.
	12690432	$\langle\!\langle 7,6,4,1\rangle\!\rangle$	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
	11315200	$\langle\!\langle 7,5,4,2\rangle\!\rangle$	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1357824	$\langle\!\langle 6,5,4,2,1,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	1357824	$\langle\!\langle 6,5,4,2,1,-\rangle\!\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 21.10: 7-modular decomposition matrix of block B_{92} of $\tilde{S}_{18}$
(defect 2, bar core $(3,1)$, weight 2, content $(6,5,5,2)$)

			$[\![7,5,3,2,1]\!]$ 3734016	$[\![7^2,3,1]\!]$ 1109504	$[\![8,5,3,2]\!]$ 15301632	$[\![8,6,3,1]\!]$ 3299328	$[\![8,7,3]\!]$ 4096	$[\![9,5,3,1]\!]$ 11271168	$[\![10,4,3,1]\!]$ 1419264	$[\![10,5,2,1]\!]$ 304128	$[\![10,7,1]\!]$ 135168
⋆	4096	$\langle\!\langle 17,1\rangle\!\rangle$	.	.	.	.	1	.	.	.	.
⋆	139264	$\langle\!\langle 15,3\rangle\!\rangle$	.	.	.	.	1	.	.	.	1
⋆	439296	$\langle\!\langle 14,3,1,+\rangle\!\rangle$	.	.	.	.	.	.	.	1	1
	439296	$\langle\!\langle 14,3,1,-\rangle\!\rangle$	.	.	.	.	.	.	.	1	1
⋆	1723392	$\langle\!\langle 12,3,2,1\rangle\!\rangle$	.	.	.	.	.	.	1	1	.
⋆	1244672	$\langle\!\langle 10,8\rangle\!\rangle$	.	1	.	.	.	.	.	.	1
⋆	4852224	$\langle\!\langle 10,7,1,+\rangle\!\rangle$	.	1	.	1	1	.	.	1	1
	4852224	$\langle\!\langle 10,7,1,-\rangle\!\rangle$	.	1	.	1	1	.	.	1	1
	16293888	$\langle\!\langle 10,5,2,1\rangle\!\rangle$	.	.	.	1	.	1	1	1	.
⋆	12690432	$\langle\!\langle 10,4,3,1\rangle\!\rangle$	.	.	.	.	.	1	1	.	.
	29872128	$\langle\!\langle 9,5,3,1\rangle\!\rangle$	.	.	1	1	.	1	.	.	.
	8146944	$\langle\!\langle 8,7,3,+\rangle\!\rangle$	1	1	.	1	1	.	.	.	.
	8146944	$\langle\!\langle 8,7,3,-\rangle\!\rangle$	1	1	.	1	1	.	.	.	.
	28288000	$\langle\!\langle 8,6,3,1\rangle\!\rangle$	2	2	1	1	.	.	.	.	.
⋆	19035648	$\langle\!\langle 8,5,3,2\rangle\!\rangle$	1	.	1	.	.	.	.	.	.
⋆	3734016	$\langle\!\langle 7,5,3,2,1,+\rangle\!\rangle$	1	.	.	.	.	.	.	.	.
	3734016	$\langle\!\langle 7,5,3,2,1,-\rangle\!\rangle$	1	.	.	.	.	.	.	.	.

Table 21.11: 7-modular decomposition matrix of block b_{50} of $\tilde{A}_{18}$
(defect 2, bar core $(3,1)$, weight 2, content $(6,5,5,2)$)

			⟦7,5,3,2,1,+⟧ 1867008	⟦7,5,3,2,1,−⟧ 1867008	⟦7^2,3,1,+⟧ 554752	⟦7^2,3,1,−⟧ 554752	⟦8,5,3,2,+⟧ 7650816	⟦8,5,3,2,−⟧ 7650816	⟦8,6,3,1,+⟧ 1649664	⟦8,6,3,1,−⟧ 1649664	⟦8,7,3,+⟧ 2048	⟦8,7,3,−⟧ 2048	⟦9,5,3,1,+⟧ 5635584	⟦9,5,3,1,−⟧ 5635584	⟦10,4,3,1,+⟧ 709632	⟦10,4,3,1,−⟧ 709632	⟦10,5,2,1,+⟧ 152064	⟦10,5,2,1,−⟧ 152064	⟦10,7,1,+⟧ 67584	⟦10,7,1,−⟧ 67584
⋆	2048	⟪17,1,+⟫	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
⋆	2048	⟪17,1,−⟫	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
⋆	69632	⟪15,3,+⟫	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.
⋆	69632	⟪15,3,−⟫	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1
⋆	439296	⟪14,3,1⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
⋆	861696	⟪12,3,2,1,+⟫	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
⋆	861696	⟪12,3,2,1,−⟫	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.
⋆	622336	⟪10,8,+⟫	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
⋆	622336	⟪10,8,−⟫	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1
⋆	4852224	⟪10,7,1⟫	.	.	1	1	.	.	1	1	1	1	.	.	.	.	1	1	1	1
⋆	8146944	⟪10,5,2,1,+⟫	.	.	.	.	.	.	.	1	.	.	1	.	.	1	1	.	.	.
	8146944	⟪10,5,2,1,−⟫	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	1	.	.
⋆	6345216	⟪10,4,3,1,+⟫	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.
⋆	6345216	⟪10,4,3,1,−⟫	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
⋆	14936064	⟪9,5,3,1,+⟫	.	.	.	.	.	1	.	1	.	.	1	.	.	.	.	.	.	.
	14936064	⟪9,5,3,1,−⟫	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.
	8146944	⟪8,7,3⟫	1	1	1	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.
⋆	14144000	⟪8,6,3,1,+⟫	1	1	1	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.
	14144000	⟪8,6,3,1,−⟫	1	1	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.
⋆	9517824	⟪8,5,3,2,+⟫	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
⋆	9517824	⟪8,5,3,2,−⟫	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
⋆	3734016	⟪7,5,3,2,1⟫	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

A The Brauer trees

The p-modular Brauer trees of $\tilde{S}_n$ and $\tilde{A}_n$ are known by [54]. Due to the full generality of the result, one has to admit some automorphisms of the underlying ordinary character tables that flip certain conjugacy classes. Hence we had to check whether or not these automorphisms are tolerated by our fixed arrangement of conjugacy classes. We did this by computing the irreducible Brauer characters of the blocks in question (which determine the corresponding Brauer tree) using the methods demonstrated in this work. We give all Brauer trees of spin p-blocks of $\tilde{S}_n$ and $\tilde{A}_n$ for $n \in \{14, \ldots, 18\}$ and $p \in \{3, 5, 7\}$.

Figure A.1: $\tilde{S}_{14}$ mod 5: Brauer tree of B_{23}

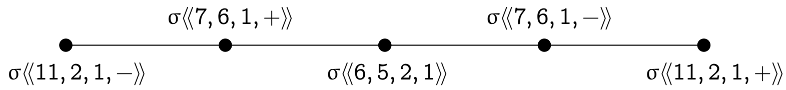


Figure A.2: $\tilde{S}_{14}$ mod 5: Brauer tree of B_{24}

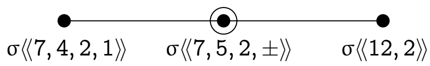


Figure A.3: $\tilde{A}_{14}$ mod 5: Brauer tree of b_{14}

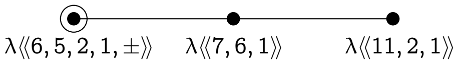


Figure A.4: $\tilde{A}_{14}$ mod 5: Brauer tree of b_{15}

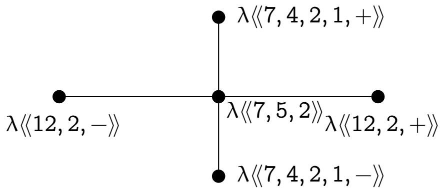


Figure A.5: $\tilde{S}_{14}$ mod 7: Brauer tree of B_{57}

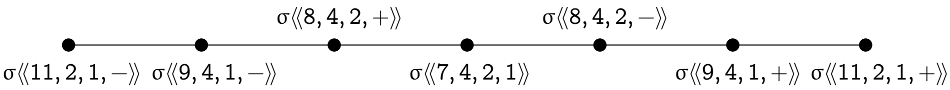


Figure A.6: $\tilde{A}_{14}$ mod 7: Brauer tree of b_{30}

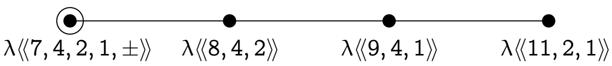


Figure A.7: $\tilde{S}_{15}$ mod 3: Brauer tree of B_9

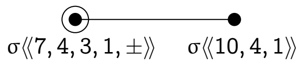


Figure A.8: $\tilde{A}_{15}$ mod 3: Brauer tree of b_6

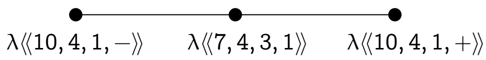


Figure A.9: $\tilde{S}_{15}$ mod 5: Brauer tree of B_{28}

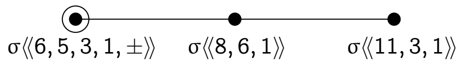


Figure A.10: $\tilde{S}_{15}$ mod 5: Brauer tree of B_{29}

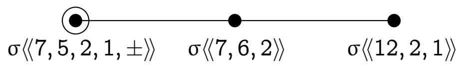


Figure A.11: $\tilde{A}_{15}$ mod 5: Brauer tree of b_{17}

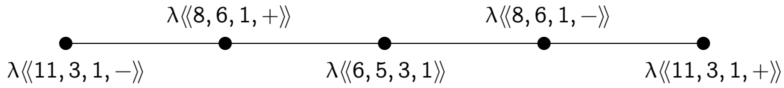


Figure A.12: $\tilde{A}_{15}$ mod 5: Brauer tree of b_{18}

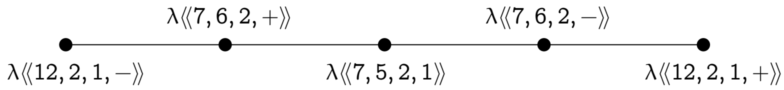


Figure A.13: $\tilde{S}_{15}$ mod 7: Brauer tree of B_{54}

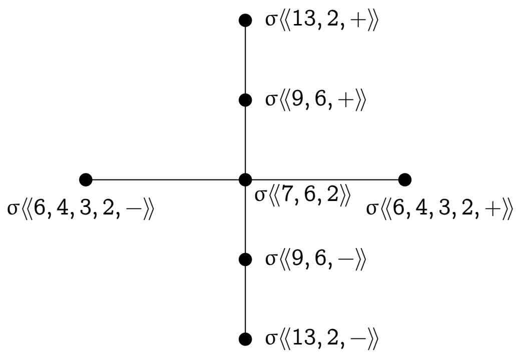


Figure A.14: $\tilde{S}_{15}$ mod 7: Brauer tree of B_{55}

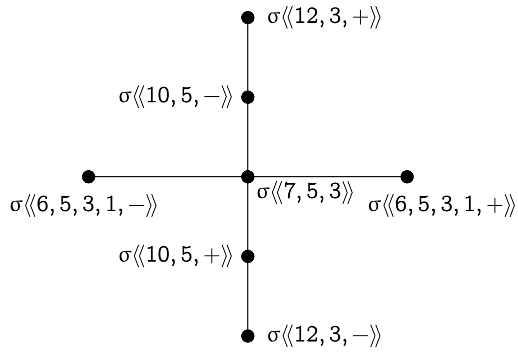


Figure A.15: $\tilde{A}_{15}$ mod 7: Brauer tree of b_{29}

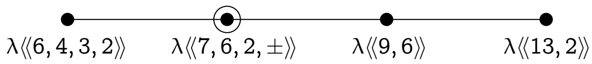


Figure A.16: $\tilde{A}_{15}$ mod 7: Brauer tree of b_{30}

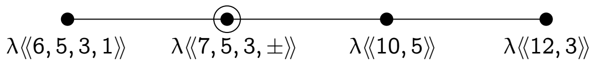


Figure A.17: $\tilde{S}_{16}$ mod 5: Brauer tree of B_{33}

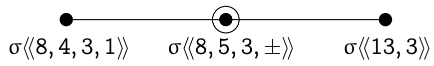


Figure A.18: $\tilde{A}_{16}$ mod 5: Brauer tree of b_{23}

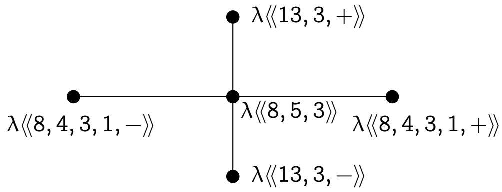


Figure A.19: $\tilde{S}_{16}$ mod 7: Brauer tree of B_{69}

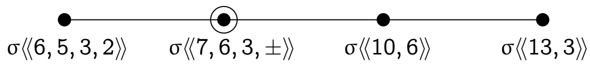


Figure A.20: $\tilde{S}_{16}$ mod 7: Brauer tree of B_{70}

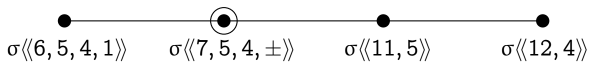


Figure A.21: $\tilde{S}_{16}$ mod 7: Brauer tree of B_{71}

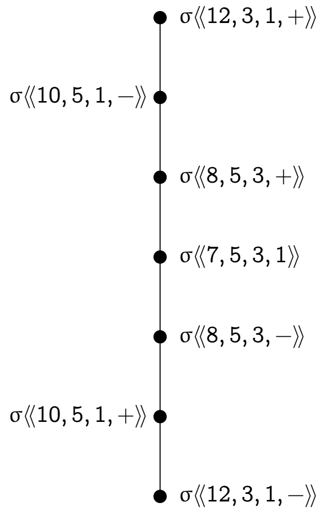


Figure A.22: $\tilde{S}_{16}$ mod 7: Brauer tree of B_{72}

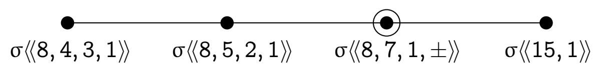


Figure A.23: $\tilde{A}_{16}$ mod 7: Brauer tree of b_{41}

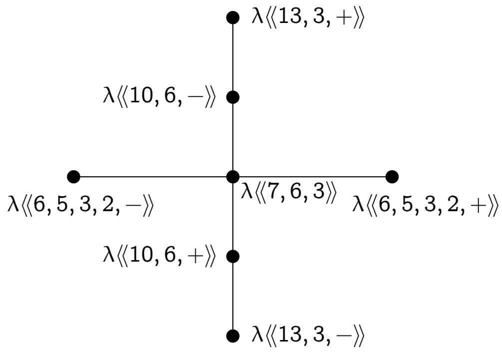


Figure A.24: $\tilde{A}_{16}$ mod 7: Brauer tree of b_{42}

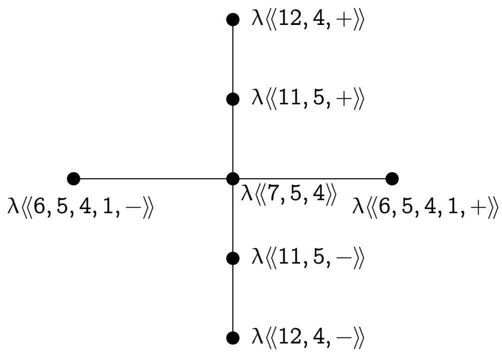


Figure A.25: $\tilde{A}_{16}$ mod 7: Brauer tree of b_{43}

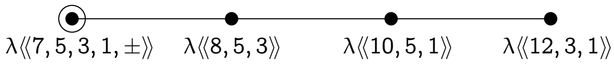


Figure A.26: $\tilde{A}_{16}$ mod 7: Brauer tree of b_{44}

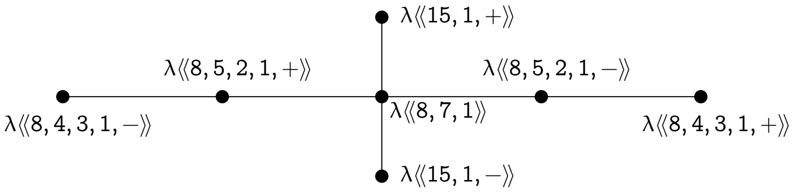


Figure A.27: $\tilde{S}_{17}$ mod 5: Brauer tree of B_{30}

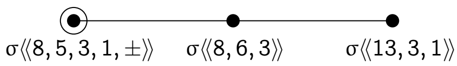


Figure A.28: $\tilde{A}_{17}$ mod 5: Brauer tree of b_{20}

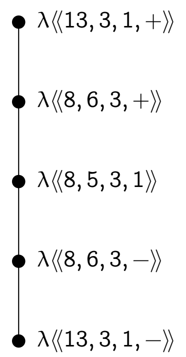


Figure A.29: $\tilde{S}_{17}$ mod 7: Brauer tree of B_{71}

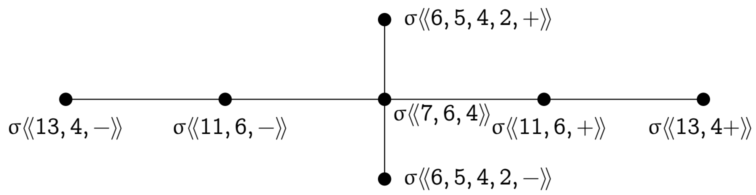


Figure A.30: $\tilde{S}_{17}$ mod 7: Brauer tree of B_{73}

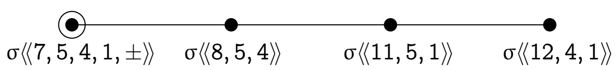


Figure A.31: $\tilde{A}_{17}$ mod 7: Brauer tree of b_{39}

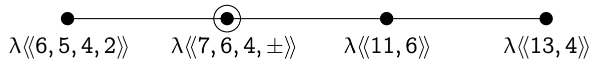


Figure A.32: $\tilde{A}_{17}$ mod 7: Brauer tree of b_{41}

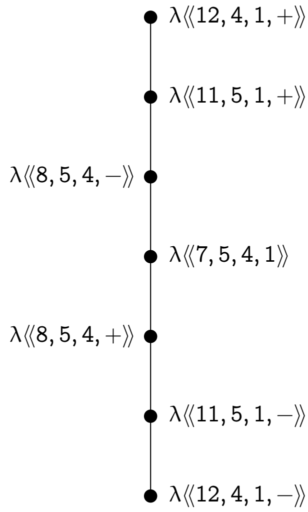


Figure A.33: $\tilde{S}_{18}$ mod 3: Brauer tree of B_{10}

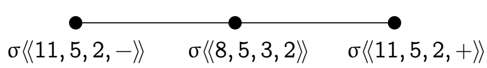


Figure A.34: $\tilde{A}_{18}$ mod 3: Brauer tree of b_7

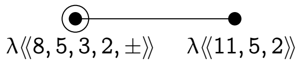


Figure A.35: $\tilde{S}_{18}$ mod 5: Brauer tree of B_{39}

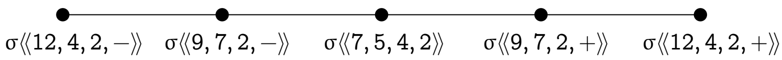


Figure A.36: $\tilde{S}_{18}$ mod 5: Brauer tree of B_{41}

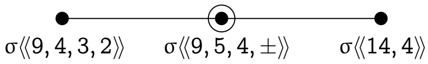


Figure A.37: $\tilde{A}_{18}$ mod 5: Brauer tree of b_{22}

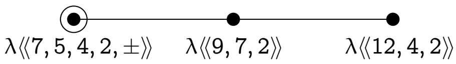


Figure A.38: $\tilde{A}_{18}$ mod 5: Brauer tree of b_{25}

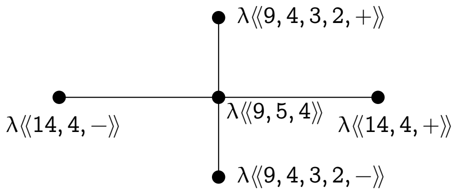


Figure A.39: $\tilde{S}_{18}$ mod 7: Brauer tree of B_{91}

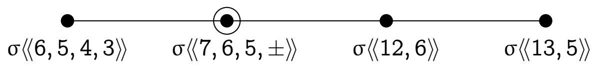


Figure A.40: $\tilde{S}_{18}$ mod 7: Brauer tree of B_{93}

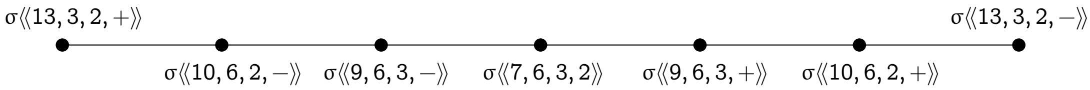


Figure A.41: $\tilde{S}_{18}$ mod 7: Brauer tree of B_{94}

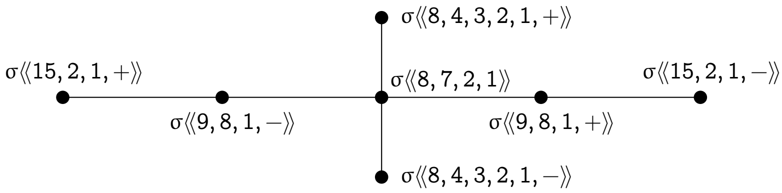


Figure A.42: $\tilde{S}_{18}$ mod 7: Brauer tree of B_{96}

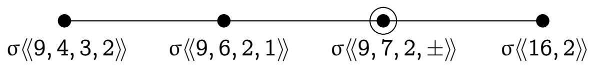


Figure A.43: $\tilde{A}_{18}$ mod 7: Brauer tree of b_{49}

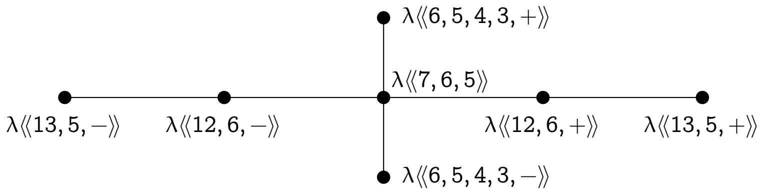


Figure A.44: $\tilde{A}_{18}$ mod 7: Brauer tree of b_{51}

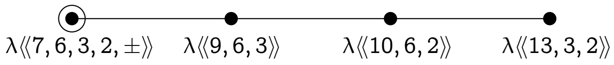


Figure A.45: $\tilde{A}_{18}$ mod 7: Brauer tree of b_{52}

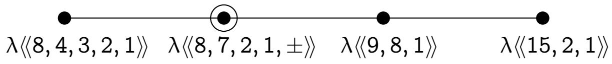
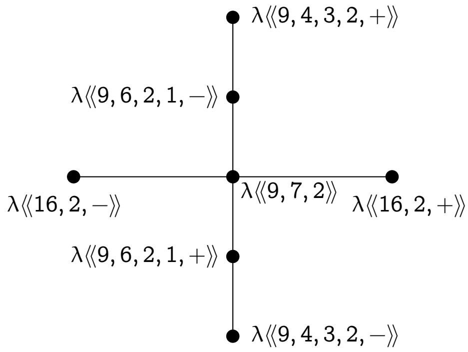


Figure A.46: $\tilde{A}_{18}$ mod 7: Brauer tree of b_{55}



B Character tables from the GAP Character Table Library

We have used the p-modular character tables of S_n for $n \leqslant 17$ and A_n for $n \leqslant 14$ that are available from the GAP Character Table Library (CtblLib) [4]. The p-modular character tables of A_n for $15 \leqslant n \leqslant 17$ can be computed easily using the methods of this work. It was necessary to identify the conjugacy classes of the table in question in compliance with our notion of standard representatives $s_\lambda := t_\lambda^\pi$, see Section 5. The necessary adjustments are described here.

It is straightforward to identify the columns of a character table of a symmetric group S_n, as there is exactly one conjugacy class (or column) corresponding to each cycle type $\lambda \in \mathcal{P}(n)$ which is stored as a class parameter on the table in question. In the case of an alternating group A_n, problems arise owing to the pairs of conjugacy classes of cycle type $\lambda \in \mathcal{O}(n) \cap \mathcal{D}^+(n)$ that fuse in S_n. To illustrate this, consider the 3-modular character table of A_{12} from the GAP Character Table Library. The classes in question, 11a, 11b, 35a, and 35b, are labeled by $(11,1)^+$, $(11,1)^-$, $(7,5)^+$, and $(7,5)^-$, respectively. There is one pair of conjugate (and dual) characters, φ_6 and φ_7, each of which has different values on each of the pairs $\{11a, 11b\}$ and $\{35a, 35b\}$. These values are (in ATLAS notation [12]) b11, /b11 and $-2 + \text{b}35$, $/(-2 + \text{b}35)$, respectively, for φ_6, and interchanged by complex conjugation (/) for its dual χ_7. We construct a 126-dimensional irreducible 3-modular representation of A_{12} which affords $\varphi \in \{\varphi_6, \varphi_7\}$, and compute the Brauer character values $\varphi(s_{(11,1)^+}) = /\text{b}11$ and $\varphi(s_{(7,5)^+}) = -2 + \text{b}35$. Thus, either $s_{(11,1)^+}$ is contained in 11a and $s_{(7,5)^+}$ in 35b, or $s_{(11,1)^+}$ is an element of 11b and $s_{(7,5)^+}$ is contained in 35a. After checking that this is consistent with the p-modular character tables of A_{12} for $p \in \{5, 7, 11\}$, we may choose to interchange the labels of 35a and 35b, and leave the ones for 11a and 11b unchanged.

Similarly, it turns out that we need to interchange the labels of 33a and 33b of A_{14}, affecting the 5- and 7-modular tables. In all other cases, the class labelling of the tables of A_n from the GAP Character Table Library is consistent with our definition.

The situation gets more complicated for character tables of $\tilde{S}_n$ or $\tilde{A}_n$. Therefore, we have chosen only to handle the character tables of alternating groups by direct inspection, and then recalculate, in a uniform setting, the p-modular spin characters of $\tilde{S}_n$ and $\tilde{A}_n$ for

$n \leqslant 13$ and $p \in \{3, 5, 7\}$. This also served as a valuable test for our own GAP programs. The uniform setting is provided by the generic construction of the ordinary character tables of $\tilde{S}_n$ and $\tilde{A}_n$, see [56], which is then inherited to the p-modular character table.

Bibliography

[1] G. E. Andrews, C. Bessenrodt, and J. B. Olsson, *Partition identities and labels for some modular characters*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 597–615.

[2] ———, *A refinement of a partition identity and blocks of some modular characters*, Arch. Math. (Basel) **66** (1996), no. 2, 101–113.

[3] C. Bessenrodt, A. O. Morris, and J. B. Olsson, *Decomposition matrices for spin characters of symmetric groups at characteristic* 3, J. Algebra **164** (1994), no. 1, 146–172.

[4] T. Breuer, *CTblLib. The GAP Character Table Library, v1.1.3.* `http://www.gap-system.org/Packages/ctbllib.html`.

[5] O. Brunat, *Basic sets in defining characteristic for general linear groups of small rank*, J. Pure Appl. Algebra **213** (2009), no. 5, 698–710.

[6] O. Brunat and J.-B. Gramain, *A 2-basic set of the alternating group*, Arch. Math. (Basel) **94** (2010), no. 4, 301–309.

[7] ———, *A basic set for the alternating group*, J. Reine Angew. Math. **641** (2010), 177–202.

[8] J. Brundan and A. Kleshchev, *Hecke-Clifford superalgebras, crystals of type* $A_{2l}^{(2)}$ *and modular branching rules for* $\hat{S}_n$, Represent. Theory **5** (2001), 317–403.

[9] ———, *Projective representations of symmetric groups via Sergeev duality*, Math. Z. **239** (2002), no. 1, 27–68.

[10] ———, *James' regularization theorem for double covers of symmetric groups*, J. Algebra **306** (2006), no. 1, 128–137.

[11] M. Cabanes, *Local structure of the* p*-blocks of* $\tilde{S}_n$, Math. Z. **198** (1988), no. 4, 519–543.

[12] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[13] C. W. Curtis and I. Reiner, *Methods of representation theory. Vol. I*, John Wiley & Sons Inc., New York, 1981.

[14] J. D. Dixon, *Exact solution of linear equations using* p*-adic expansions*, Numer. Math. **40** (1982), no. 1, 137–141.

[15] M. Fayers, *James's Conjecture holds for weight four blocks of Iwahori-Hecke algebras*, J. Algebra **317** (2007), no. 2, 593–633.

[16] D. M. Goldschmidt, *Lectures on character theory*, Publish or Perish Inc., Wilmington, Del., 1980.

[17] R. Gow, B. Huppert, R. Knörr, O. Manz, and W. Willems, *Representation theory in arbitrary characteristic*, Casa Editrice Dott. Antonio Milani (CEDAM), Padua, 1993.

[18] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008. `http://www.gap-system.org`.

[19] G. Hiss, C. Jansen, K. Lux, and R. Parker, *Computational modular character theory*, 1993. `http://www.math.rwth-aachen.de/~moc/CoMoChaT/`.

[20] G. Hiss and K. Lux, *Brauer trees of sporadic groups*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1989.

[21] P. N. Hoffman and J. F. Humphreys, *Projective representations of the symmetric groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1992.

[22] D. F. Holt and S. Rees, *Testing modules for irreducibility*, J. Austral. Math. Soc. Ser. A **57** (1994), no. 1, 1–16.

[23] J. F. Humphreys, *Blocks of projective representations of the symmetric groups*, J. London Math. Soc. (2) **33** (1986), no. 3, 441–452.

[24] I. M. Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976.

[25] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[26] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.

[27] C. Jansen, *Ein Atlas* 3*-modularer Charaktertafeln*, Dissertation, RWTH Aachen, M + M Wissenschaftsverlag, Krefeld (1995).

[28] C. Jansen, K. Lux, R. Parker, and R. Wilson, *An atlas of Brauer characters*, London Mathematical Society Monographs. New Series, vol. 11, The Clarendon Press Oxford University Press, New York, 1995.

[29] T. Józefiak, *Semisimple superalgebras.*, Algebra: some current trends, Lect. Notes Math. 1352 (1988), 96–113.

[30] ———, *Characters of projective representations of symmetric groups*, Exposition. Math. **7** (1989), no. 3, 193–247.

[31] R. Kessar, *Blocks and source algebras for the double covers of the symmetric and alternating groups.*, J. Algebra **186** (1996), no. 3, 872–933.

[32] A. Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics, vol. 163, Cambridge University Press, Cambridge, 2005.

[33] A. Kleshchev and V. Shchigolev, *Modular branching rules for projective representations of symmetric groups and lowering operators for the supergroup* $Q(n)$, Mem. Amer. Math. Soc. (to appear in print), arXiv:1011.0566v1 (2010).

[34] A. S. Kleshchev and P. H. Tiep, *On restrictions of modular spin representations of symmetric and alternating groups*, Trans. Amer. Math. Soc. **356** (2004), no. 5, 1971–1999 (electronic).

[35] ———, *Small dimension projective representations of symmetric and alternating groups*, Preprint, arXiv:1106.3123v1 (201106), available at `1106.3123v1`.

[36] B. Leclerc and J.-Y. Thibon, q-*deformed Fock spaces and modular representations of spin symmetric groups*, J. Phys. A **30** (1997), no. 17, 6163–6176.

[37] F. Lübeck, *Conway polynomials for finite fields*, `http://www.math.rwth-aachen.de/~Frank.Luebeck/data/ConwayPol/`.

[38] K. Lux, *Algorithmic Methods in Modular Representation Theory*, Habilitationsschrift, RWTH Aachen (1997).

[39] K. Lux, J. Müller, and M. Ringe, *Peakword condensation and submodule lattices: an application of the MEAT-AXE*, J. Symbolic Comput. **17** (1994), no. 6, 529–544.

[40] K. Lux and H. Pahlings, *Computational aspects of representation theory of finite groups*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 1991, pp. 37–64.

[41] ———, *Computational aspects of representation theory of finite groups II*, Algorithmic algebra and number theory (Heidelberg, 1997), 1999, pp. 381–397.

[42] ———, *Representations of groups*, Cambridge Studies in Advanced Mathematics, vol. 124, Cambridge University Press, Cambridge, 2010.

[43] K. Lux and M. Wiegelmann, *Condensing tensor product modules*, The atlas of finite groups: ten years on (Birmingham, 1995), 1998, pp. 174–190.

[44] ———, *Determination of socle series using the condensation method*, J. Symbolic Comput. **31** (2001), no. 1-2, 163–178.

[45] L. Maas, *On a construction of the basic spin representations of symmetric groups*, Comm. Algebra **38** (2010), no. 12, 4545–4552.

[46] A. Mathas, *Specht 2.4*. `http://www.maths.usyd.edu.au/u/mathas/specht/`.

[47] The Modular Atlas project. http://www.math.rwth-aachen.de/~MOC/.

[48] A. O. Morris, *The spin characters of the symmetric groups*, Quart. J. Math. Oxford Ser. (2) **13** (1962), 241–246.

[49] ———, *The spin representation of the symmetric group*, Proc. London Math. Soc. (3) **12** (1962), 55–76.

[50] ———, *The spin characters of the symmetric group. II*, Quart. J. Math. Oxford Ser. (2) **14** (1963), 247–253.

[51] ———, *The spin representation of the symmetric group*, Canad. J. Math. **17** (1965), 543–549.

[52] A. O. Morris and A. K. Yaseen, *Decomposition matrices for spin characters of symmetric groups*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 1-2, 145–164.

[53] J. Müller, *private communication.*

[54] ———, *Brauer trees for the Schur cover of the symmetric group*, J. Algebra **266** (2003), no. 2, 427–445.

[55] W. Nickel, *Endliche Körper in dem gruppentheoretischen Programmsystem GAP*, Diplomarbeit, RWTH Aachen (1988).

[56] F. Noeske, *Zur Darstellungstheorie der Schurschen Erweiterungen symmetrischer Gruppen*, Diplomarbeit, RWTH Aachen (2002).

[57] ———, *Morita-Äquivalenzen in der algorithmischen Darstellungstheorie*, Dissertation, RWTH Aachen (2005).

[58] ———, *Tackling the generation problem in condensation*, J. Algebra **309** (2007), no. 2, 711–722.

[59] J. B. Olsson, *On the* p*-blocks of symmetric and alternating groups and their covering groups*, J. Algebra **128** (1990), no. 1, 188–213.

[60] ———, *The number of modular characters in certain* p*-blocks*, Proc. London Math. Soc. (3) **65** (1992), no. 2, 245–264.

[61] R. A. Parker, *The computer calculation of modular characters (the meat-axe)*, Computational group theory (Durham, 1982), 1984, pp. 267–274.

[62] M. Ringe, The MeatAxe 2.4.13, `http://www.math.rwth-aachen.de/~MTX/` (2009).

[63] R. L. Roth, *A dual view of the Clifford theory of characters of finite groups*, Canad. J. Math. **23** (1971), 857–865.

[64] A. J. E. Ryba, *Computer condensation of modular representations*, J. Symbolic Comput. **9** (1990), no. 5-6, 591–600.

[65] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.*, J. Reine Angew. Math. **127** (1904), 20–50.

[66] ———, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.*, J. Reine Angew. Math. **132** (1907), 85–137.

[67] ———, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.

[68] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134.

[69] D. B. Wales, *Some projective representations of* S_n, J. Algebra **61** (1979), no. 1, 37–57.

[70] R. A. Wilson, R. A. Parker, S. Nickerson, J. N. Bray, and T. Breuer, *AtlasRep. A GAP Interface to the Atlas of Group Representations, v1.4.0.* `http://www.gap-system.org/Packages/atlasrep.html`.

[71] A. K. Yaseen, *Modular spin representations of the symmetric group*, Ph.D. thesis, The University College of Wales, Aberystwyth (1987).